For an excellent discussion of both the mathematical problems and experiments with soap films, we refer to Courant and Robbins in Additional Readings. The larger message of this subsection is the extent to which one must use caution in accepting solutions of the Euler equations.

Exercises

22.1.1 For $dy/dx \equiv y_x \neq 0$, show the equivalence of the two forms of Euler's equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$$

and

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0.$$

22.1.2 Derive Euler's equation by expanding the integrand of

$$J(\alpha) = \int_{x_1}^{x_2} f\Big(y(x,\alpha), y_x(x,\alpha), x\Big)\, dx$$

in powers of α.

Note. The stationary condition is $\partial J(\alpha)/\partial\alpha = 0$, evaluated at $\alpha = 0$. The terms quadratic in α may be useful in establishing the nature of the stationary solution (maximum, minimum, or saddle point).

22.1.3 Find the Euler equation corresponding to Eq. (22.14) if $f = f(y_{xx}, y_x, y, x)$, assuming that y and y_x have fixed values at the endpoints of their interval of definition.

ANS. $\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y_{xx}}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) + \frac{\partial f}{\partial y} = 0.$

22.1.4 The integrand $f(y, y_x, x)$ of Eq. (22.2) has the form

$$f(y, y_x, x) = f_1(x, y) + f_2(x, y)y_x.$$

(a) Show that the Euler equation leads to

$$\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = 0.$$

(b) What does this imply for the dependence of the integral J on the choice of path?

22.1.5 Show that the condition that $J = \int f(x, y)\, dx$ has a stationary value

(a) leads to $f(x, y)$ independent of y and

(b) yields no information about any x-dependence.

We get no (continuous, differentiable) solution. To be a meaningful variational problem, dependence on y or higher derivatives is essential.

Note. The situation will change when constraints are introduced (compare to Exercise 22.4.6).

22.1.6 A soap film stretched between two rings of unit radius centered at $\pm x_0$ will have its closest approach to the x-axis at $x = 0$, with the distance from the axis given by c_1, with x_0 and c_1 related by Eq. (22.26) or Eq. (22.29).

(a) Show that dc_1/dx_0 becomes infinite when $x_0 \sinh(x_0/c_1) = 1$, indicating that the soap film becomes unstable if x_0 is increased beyond the value satisfying this condition.

(b) Show that the condition of part (a) is equivalent to

$$\frac{x_0}{c_1} = \coth\left(\frac{x_0}{c_1}\right).$$

(c) Solve the transcendental equation of part (b) to obtain the critical value of x_0/c_1 and show that the separate values of x_0 and c_1 are then approximately $x_0 \approx 0.6627$ and $c_1 \approx 0.5524$.

22.1.7 A soap film is stretched across the space between two rings of unit radius centered at $\pm x_0$ on the x-axis and perpendicular to the x-axis. Using the solution developed in Example 22.1.3, set up the transcendental equations for the condition that x_0 is such that the area of the curved surface of rotation equals the area of the two rings (Goldschmidt discontinuous solution). Solve for x_0.

22.1.8 In Example 22.1.1, expand $J[y(x,\alpha)] - J[y(x,0)]$ in powers of α. The term linear in α leads to the Euler equation and to the straight-line solution, Eq. (22.16). Investigate the α^2 term and show that the stationary value of J, the straight-line distance, is a **minimum**.

22.1.9 (a) Show that the integral

$$J = \int_{x_1}^{x_2} f(y, y_x, x)\, dx, \quad \text{with} \quad f = y(x),$$

has **no** extreme values.

(b) If $f(y, y_x, x) = y^2(x)$, find a discontinuous solution similar to the Goldschmidt solution for the soap-film problem.

22.1.10 Fermat's principle of optics states that a light ray in a medium for which n is the (position-dependent) index of refraction will follow the path $y(x)$ for which

$$\int_{x_1,y_1}^{x_2,y_2} n(y,x)\, ds$$

is a minimum. For $y_2 = y_1 = 1$, $-x_1 = x_2 = 1$, find the ray path if

(a) $n = e^y$, (b) $n = a(y - y_0)$, $y > y_0$.

22.1.11 A particle moves, starting at rest, from point A on the surface of the Earth to point B (also on the surface) by sliding frictionlessly through a tunnel. Find the differential equation satisfied by the path if the transit time is to be a minimum. Assume the Earth to be a nonrotating sphere of uniform density.

Hint. The potential energy of a particle of mass m a distance $r < R$ from the center of the Earth, with R the Earth's radius, is $\frac{1}{2}mg(R^2 - r^2)/R$, where g is the gravitational acceleration at the Earth's surface. It is convenient to describe the path of the particle (in the plane through A, B, and the center of the Earth) by plane polar coordinates (r, θ), with A at $(R, -\varphi)$ and B at (R, φ).

ANS. Letting r_0 be the minimum value of r (reached at $\theta = 0$), Eq. (22.21) yields $r_\theta^2 = \dfrac{r^2 R^2 (r^2 - r_0^2)}{r_0^2 (R^2 - r^2)}$ (the constant in that equation has the value such that $r_\theta = 0$ at $\theta = 0$).

The solution for the path is a hypocycloid, generated by a circle of radius $\frac{1}{2}(R - r_0)$ rolling inside the circle of radius R. You might like to show that the transit time is

$$t = \pi \frac{(R^2 - r_0^2)^{1/2}}{(Rg)^{1/2}}.$$

For details see P. W. Cooper, *Am. J. Phys.* **34**: 68 (1966); G. Veneziano, *et al.,* **34**: 701 (1966).

22.1.12 A ray of light follows a straight-line path in a first homogeneous medium, is refracted at an interface, and then follows a new straight-line path in the second medium. See Fig. 22.7. Use Fermat's principle of optics to derive Snell's law of refraction:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Hint. Keep the points (x_1, y_1) and (x_2, y_2) fixed and vary x_0 to satisfy Fermat's principle.

Note. This is **not** an Euler equation problem, because the light path is not differentiable at x_0.

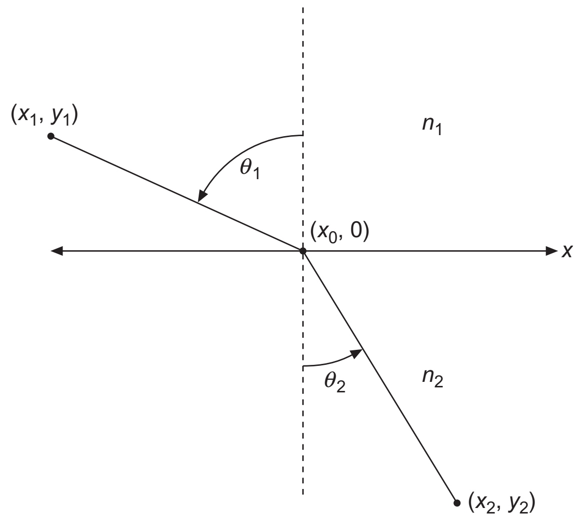

FIGURE 22.7 Snell's law.

22.1.13 A second soap-film configuration for the unit-radius rings at $x = \pm x_0$ consists of a circular disk, radius a, in the $x = 0$ plane and two catenoids of revolution, one joining the disk and each ring. One catenoid may be described by

$$y = c_1 \cosh\left(\frac{x}{c_1} + c_3\right).$$

(a) Impose boundary conditions at $x = 0$ and $x = x_0$.

(b) Although not necessary, it is convenient to require that the catenoids form an angle of 120° where they join the central disk. Express this third boundary condition in mathematical terms.

(c) Show that the total area of catenoids plus central disk is then

$$A = c_1^2\left[\sinh\left(\frac{2x_0}{c_1} + 2c_3\right) + \frac{2x_0}{c_1}\right].$$

Note. Although this soap-film configuration is physically realizable and stable, the area is larger than that of the simple catenoid for all ring separations for which both films exist.

ANS. (a) $\begin{cases} 1 = c_1 \cosh\left(\dfrac{x_0}{c_1} + c_3\right) \\ a = c_1 \cosh c_3 \end{cases}$ (b) $\dfrac{dy}{dx} = \tan 30° = \sinh c_3.$

22.1.14 For the soap film described in Exercise 22.1.13, find (numerically) the maximum value of x_0.

Note. This calls for a calculator with hyperbolic functions or a table of hyperbolic cotangents.

ANS. $x_{0\,\max} = 0.4078.$

22.1.15 Find the curve of quickest descent from $(0, 0)$ to (x_0, y_0) for a particle that, starting from rest, slides under gravity and without friction. Show that the ratio of times taken by the particle along a straight line joining the two points compared to along the curve of quickest descent is $(1 + 4/\pi^2)^{1/2}$.

Hint. Take y to increase downwards. Apply Eq. (22.21) to obtain $y_x^2 = (1 - c^2 y)/c^2 y$, where c is an integration constant. It is helpful to make the substitution $c^2 y = \sin^2 \varphi/2$ and take $(x_0, y_0) = (\pi/2c^2, 1/c^2)$.

22.2 More General Variations

Several Dependent Variables

To apply variational methods to classical mechanics, we need to generalize the Euler equation to situations in which there is more than one dependent variable in roles like y in