and that for all x, $\cosh x \geq 1$, we see that $\sinh z$ is zero only for $z = n\pi i$, with n an integer. Moreover, because $\lim_{z\to 0} z/\sinh z = 1$, the integrand of our present contour integral will not have a pole at $z = 0$, but will have poles at $z = n\pi i$ for all nonzero integral n. For that reason, the lower horizontal line of the contour in Fig. 11.27, marked A, continues through $z = 0$ as a straight line on the real axis, but the upper horizontal line (for which $y = \pi$), marked B and B', has an infinitesimal semicircular detour, marked C, around the pole at $z = \pi i$.

Because the integrand in Eq. (11.120) is an even function of z, the integral on segment A, which extends from $-\infty$ to $+\infty$, has the value $2I$. To evaluate the integral on segments B and B', we first note, using Eq. (11.121), that $\sinh(x + i\pi) = -\sinh x$, and that the integral on these segments is in the direction of negative x. Recognizing the integral on these segments as a Cauchy principal value, we write

$$\int_{B+B'} \frac{z\,dz}{\sinh z} = \fint_{-\infty}^{\infty} \frac{x + i\pi}{\sinh x} dx.$$

Because $x/\sinh x$ is even and nonsingular at $z = 0$, while $i\pi/\sinh x$ is odd, this integral reduces to

$$\fint_{-\infty}^{\infty} \frac{x + i\pi}{\sinh x} dx = 2I.$$

Combining what we have up to this point, invoking the residue theorem, and noting that the integrand is negligible on the vertical connections at $x = \pm\infty$. We have

$$\oint \frac{z\,dz}{\sinh z} = 4I + \int_C \frac{z\,dz}{\sinh z} = 2\pi i \text{ (residue of } z/\sinh z \text{ at } z = \pi i). \tag{11.122}$$

To complete the evaluation, we now note that the residue we need is

$$\lim_{z\to\pi i} \frac{z(z - \pi i)}{\sinh z} = \frac{\pi i}{\cosh \pi i} = -\pi i,$$

and, cf. Eqs. (11.75) and (11.76), the counterclockwise semicircle C evaluates to πi times this residue. We have then

$$4I + (\pi i)(-\pi i) = (2\pi i)(-\pi i), \quad \text{so } I = \frac{\pi^2}{4}.$$

■

Exercises

11.8.1 Generalizing Example 11.8.1, show that

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2 - b^2)^{1/2}}, \quad \text{for } a > |b|.$$

What happens if $|b| > |a|$?

11.8.2 Show that $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}, \quad a > 1.$

11.8.3 Show that $\int_0^{2\pi} \frac{d\theta}{1-2t\cos\theta+t^2} = \frac{2\pi}{1-t^2}, \quad \text{for } |t| < 1.$

What happens if $|t| > 1$? What happens if $|t| = 1$?

11.8.4 Evaluate $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}.$

ANS. $\pi/12.$

11.8.5 With the calculus of residues, show that

$$\int_0^{\pi} \cos^{2n}\theta \, d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)!!}, \quad n = 0, 1, 2, \ldots .$$

The double factorial notation is defined in Eq. (1.76).

Hint. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}), \quad |z| = 1.$

11.8.6 Verify that simplification of the expression in Eq. (11.112) yields the result given in Eq. (11.113).

11.8.7 Complete the details of Example 11.8.8 by verifying that there is no contribution to the contour integral from either the small or the large circles of the contour, and that Eq. (11.115) simplifies to the result given as (11.116).

11.8.8 Evaluate $\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2} dx, \quad a > b > 0.$

ANS. $\pi(a-b).$

11.8.9 Prove that $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$

Hint. $\sin^2 x = \frac{1}{2}(1 - \cos 2x).$

11.8.10 Show that $\int_0^{\infty} \frac{x \sin x}{x^2+1} dx = \frac{\pi}{2e}.$

11.8.11 A quantum mechanical calculation of a transition probability leads to the function $f(t,\omega) = 2(1-\cos\omega t)/\omega^2$. Show that

$$\int_{-\infty}^{\infty} f(t,\omega)\, d\omega = 2\pi t.$$

11.8.12 Show that ($a > 0$):

(a) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi}{a} e^{-a}.$$

How is the right side modified if $\cos x$ is replaced by $\cos kx$?

(b) $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\, dx = \pi\, e^{-a}.$$

How is the right side modified if $\sin x$ is replaced by $\sin kx$?

11.8.13 Use the contour shown (Fig. 11.28) with $R \to \infty$ to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

11.8.14 In the quantum theory of atomic collisions, we encounter the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin t}{t} e^{ipt}\, dt,$$

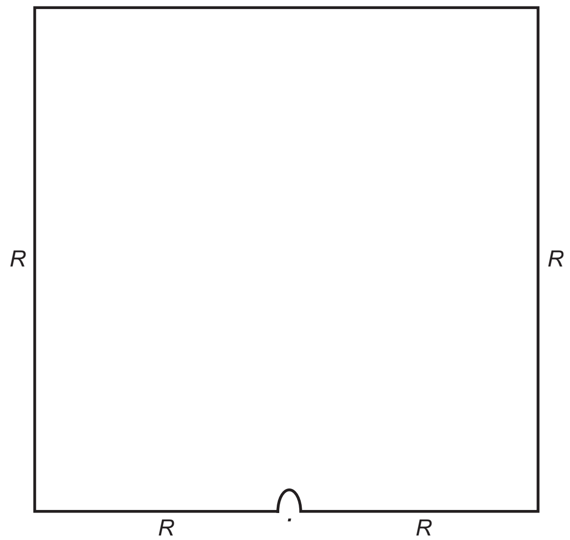

FIGURE 11.28 Contour for Exercise 11.8.13.

in which p is real. Show that

$$I = 0, \ |p| > 1$$
$$I = \pi, \ |p| < 1.$$

What happens if $p = \pm 1$?

11.8.15 Show that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a > 0.$

11.8.16 Evaluate $\displaystyle\int_{-\infty}^\infty \frac{x^2}{1+x^4}\,dx.$

ANS. $\pi/\sqrt{2}$.

11.8.17 Evaluate $\displaystyle\int_0^\infty \frac{x^p \ln x}{x^2+1}\,dx, \quad 0 < p < 1.$

ANS. $\displaystyle\frac{\pi^2}{4}\frac{\sin(\pi p/2)}{\cos^2(\pi p/2)}$.

11.8.18 Evaluate $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx,$

(a) by appropriate series expansion of the integrand to obtain

$$4\sum_{n=0}^\infty (-1)^n(2n+1)^{-3},$$

(b) and by contour integration to obtain $\dfrac{\pi^3}{8}$.

Hint. $x \to z = e^t$. Try the contour shown in Fig. 11.29, letting $R \to \infty$.

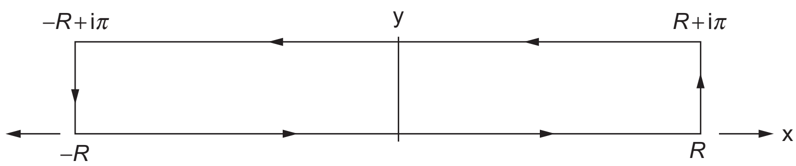

FIGURE 11.29 Contour for Exercise 11.8.18.

11.8.19 Prove that $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi \ln 2$.

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,dx = \frac{\pi a}{\sin \pi a},$$

where $-1 < a < 1$.

Hint. Use the contour shown in Fig. 11.26, noting that $z = 0$ is a branch point and the positive x-axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = 2\pi/n$.

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2 \cos 2\theta + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ($n = 4$) is a special case of this result.

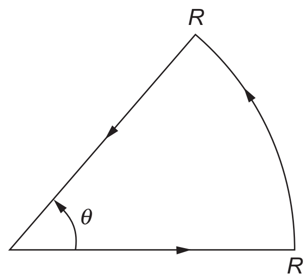

FIGURE 11.30 Sector contour.

11.8.24 Show that

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi},$$

where $0 < a < 1$.

Hint. You have a branch point and you will need a cut line. Try the contour shown in Fig. 11.26.

11.8.25 Show that $\displaystyle\int_0^\infty \frac{\cosh bx}{\cosh x}\,dx = \frac{\pi}{2\cos(\pi b/2)}, \quad |b| < 1.$

Hint. Choose a contour that encloses one pole of $\cosh z$.

11.8.26 Show that

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = \pi/4$.

Note. These are the Fresnel integrals for the special case of infinity as the upper limit. For the general case of a varying upper limit, asymptotic expansions of the Fresnel integrals are the topic of Exercise 12.6.1.

11.8.27 Show that $\displaystyle\int_0^1 \frac{1}{(x^2 - x^3)^{1/3}}\,dx = 2\pi/\sqrt{3}.$

Hint. Try the contour shown in Fig. 11.31.

11.8.28 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\tan^{-1} ax\,dx}{x(x^2+b^2)}$, for a and b positive, with $ab < 1$.

Explain why the integrand does not have a singularity at $x = 0$.

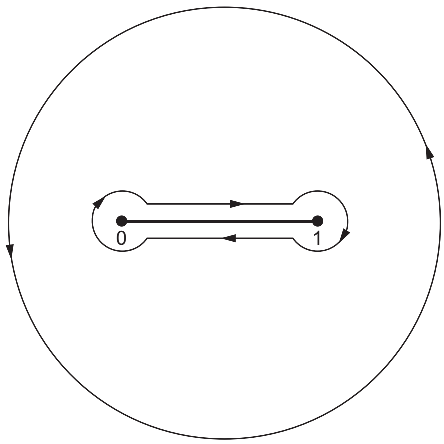

FIGURE 11.31 Contour for Exercise 11.8.27.

Hint. Try the contour shown in Fig. 11.32, and use Eq. (1.137) to represent $\tan^{-1} az$. After cancellation, the integrals on segments B and B' combine to give an elementary integral.

11.9 EVALUATION OF SUMS

The fact that the cotangent is a meromorphic function with regularly spaced poles, all with the same residue, enables us to use it to write a wide variety of infinite summations in terms of contour integrals. To start, note that $\pi \cot \pi z$ has simple poles at all integers on the real axis, each with residue

$$\lim_{z\to n} \frac{\pi \cos \pi z}{\sin \pi z} = 1.$$

Suppose that we now evaluate the integral

$$I_N = \oint_{C_N} f(z)\pi \cot \pi z \, dz,$$

where the contour is a circle about $z = 0$ of radius $N + \frac{1}{2}$ (thereby not passing close to the singularities of $\cot \pi z$). Assuming also that $f(z)$ has only isolated singularities, at points z_j other than real integers, we get by application of the residue theorem (see also Exercise 11.9.1),

$$I_N = 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum_j (\text{residues of } f(z)\pi \cot \pi z \text{ at singularities } z_j \text{ of } f).$$

This integral over the circular contour C_N will be negligible for large $|z|$ if $zf(z) \to 0$ at large $|z|$.[8] When that condition is met, $\lim_{N\to\infty} I_N = 0$, and we have the useful result

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_j (\text{residues of } f(z)\pi \cot \pi z \text{ at singularities } z_j \text{ of } f). \tag{11.123}$$

The condition required of $f(z)$ will usually be satisfied if the summation of Eq. (11.123) converges.

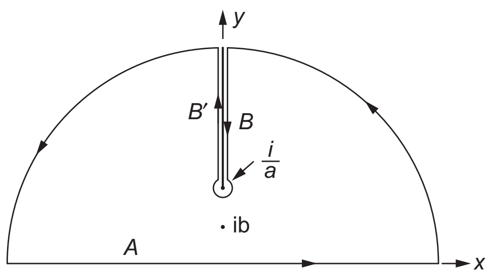

FIGURE 11.32 Contour for Exercise 11.8.28.

[8] See also Exercise 11.9.2.