are simple and at points z_n, so that μ in Eq. (11.84) is 1, we can invoke the Mittag-Leffler theorem to write f'/f as the pole expansion

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{z - z_n} + \frac{1}{z_n}\right]. \tag{11.87}$$

Integrating Eq. (11.87) yields

$$\int_0^z \frac{f'(z)}{f(z)}dz = \ln f(z) - \ln f(0)$$

$$= \frac{zf'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\ln(z - z_n) - \ln(-z_n) + \frac{z}{z_n}\right].$$

Exponentiating, we obtain the product expansion

$$f(z) = f(0)\exp\left(\frac{zf'(0)}{f(0)}\right)\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}. \tag{11.88}$$

Examples are the product expansions for

$$\sin z = z \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{z/n\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right), \tag{11.89}$$

$$\cos z = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(n - 1/2)^2\pi^2}\right). \tag{11.90}$$

The expansion of $\sin z$ cannot be obtained directly from Eq. (11.88), but its derivation is the subject of Exercise 11.7.5. We also point out here that the gamma function has a product expansion, discussed in Chapter 13.

Exercises

11.7.1 Determine the nature of the singularities of each of the following functions and evaluate the residues ($a > 0$).

(a) $\dfrac{1}{z^2 + a^2}$. (b) $\dfrac{1}{(z^2 + a^2)^2}$.

(c) $\dfrac{z^2}{(z^2 + a^2)^2}$. (d) $\dfrac{\sin 1/z}{z^2 + a^2}$.

(e) $\dfrac{ze^{+iz}}{z^2 + a^2}$. (f) $\dfrac{ze^{+iz}}{z^2 - a^2}$.

(g) $\dfrac{e^{+iz}}{z^2 - a^2}$. (h) $\dfrac{z^{-k}}{z + 1}$, $0 < k < 1$.

Hint. For the point at infinity, use the transformation $w = 1/z$ for $|z| \to 0$. For the residue, transform $f(z)dz$ into $g(w)dw$ and look at the behavior of $g(w)$.

11.7.2 Evaluate the residues at $z = 0$ and $z = -1$ of $\pi \cot \pi z / z(z+1)$.

11.7.3 The classical definition of the exponential integral $\text{Ei}(x)$ for $x > 0$ is the Cauchy principal value integral

$$\text{Ei}(x) = \fint_{-\infty}^{x} \frac{e^t}{t}\, dt,$$

where the integration range is cut at $x = 0$. Show that this definition yields a convergent result for positive x.

11.7.4 Writing a Cauchy principal value integral to deal with the singularity at $x = 1$, show that, if $0 < p < 1$,

$$\fint_{0}^{\infty} \frac{x^{-p}}{x-1}\, dx = -\pi \cot p\pi.$$

11.7.5 Explain why Eq. (11.88) is not directly applicable to the product expansion of $\sin z$. Show how the expansion, Eq. (11.89), can be obtained by expanding instead $\sin z / z$.

11.7.6 Starting from the observations

1. $f(z) = a_n z^n$ has n zeros, and
2. for sufficiently large $|R|$, $|\sum_{m=0}^{n-1} a_m R^m| < |a_n R^n|$,

use Rouché's theorem to prove the fundamental theorem of algebra (namely that every polynomial of degree n has n roots).

11.7.7 Using Rouché's theorem, show that all the zeros of $F(z) = z^6 - 4z^3 + 10$ lie between the circles $|z| = 1$ and $|z| = 2$.

11.7.8 Derive the pole expansions of $\sec z$ and $\csc z$ given in Eqs. (11.82) and (11.83).

11.7.9 Given that $f(z) = (z^2 - 3z + 2)/z$, apply a partial fraction decomposition to f'/f and show directly that $\oint_C f'(z)/f(z)\, dz = 2\pi i (N_f - P_f)$, where N_f and P_f are, respectively, the numbers of zeros and poles encircled by C (including their multiplicities).

11.7.10 The statement that the integral halfway around a singular point is equal to one-half the integral all the way around was limited to simple poles. Show, by a specific example, that

$$\int_{\text{Semicircle}} f(z)\, dz = \frac{1}{2} \oint_{\text{Circle}} f(z)\, dz$$

does not necessarily hold if the integral encircles a pole of higher order.

Hint. Try $f(z) = z^{-2}$.

11.7.11 A function $f(z)$ is analytic along the real axis except for a third-order pole at $z = x_0$. The Laurent expansion about $z = x_0$ has the form

$$f(z) = \frac{a_{-3}}{(z-x_0)^3} + \frac{a_{-1}}{z-x_0} + g(z),$$

with $g(z)$ analytic at $z = x_0$. Show that the Cauchy principal value technique is applicable, in the sense that

(a) $\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right\}$ is finite.

(b) $\int_{C_{x_0}} f(z)\,dz = \pm i\pi a_{-1}$,

where C_{x_0} denotes a **small semicircle** about $z = x_0$.

11.7.12 The unit step function is defined as (compare Exercise 1.15.13)

$$u(s-a) = \begin{cases} 0, & s < a \\ 1, & s > a. \end{cases}$$

Show that $u(s)$ has the integral representations

(a) $u(s) = \lim_{\varepsilon\to 0^+} \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ixs}}{x - i\varepsilon}\,dx.$

(b) $u(s) = \dfrac{1}{2} + \dfrac{1}{2\pi i}\displaystyle\fint_{-\infty}^{\infty} \frac{e^{ixs}}{x}\,dx.$

Note. The parameter s is real.

11.8 Evaluation of Definite Integrals

Definite integrals appear repeatedly in problems of mathematical physics as well as in pure mathematics. In Chapter 1 we reviewed several methods for integral evaluation, there noting that contour integration methods were powerful and deserved detailed study. We have now reached a point where we can explore these methods, which are applicable to a wide variety of definite integrals with physically relevant integration limits. We start with applications to integrals containing trigonometric functions, which we can often convert to forms in which the variable of integration (originally an angle) is converted into a complex variable z, with the integration integral becoming a contour integral over the unit circle.

Trigonometric Integrals, Range (0,2π)

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta, \tag{11.91}$$