Table 1.2, can be manipulated (see Chapter 13) to show that $\Gamma(z+1) = z\Gamma(z)$, consistent with the elementary result that $n! = n(n-1)!$. This functional relation can be used to analytically continue $\Gamma(z)$ to values of z for which the integral representation does not converge.

Exercises

11.6.1 As an example of an essential singularity consider $e^{1/z}$ as z approaches zero. For any complex number z_0, $z_0 \neq 0$, show that

$$e^{1/z} = z_0$$

has an infinite number of solutions.

11.6.2 Show that the function

$$w(z) = (z^2 - 1)^{1/2}$$

is single-valued if we make branch cuts on the real axis for $x > 1$ and for $x < -1$.

11.6.3 A function $f(z)$ can be represented by

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

in which $f_1(z)$ and $f_2(z)$ are analytic. The denominator, $f_2(z)$, vanishes at $z = z_0$, showing that $f(z)$ has a pole at $z = z_0$. However, $f_1(z_0) \neq 0$, $f_2'(z_0) \neq 0$. Show that a_{-1}, the coefficient of $(z - z_0)^{-1}$ in a Laurent expansion of $f(z)$ at $z = z_0$, is given by

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

11.6.4 Determine a unique branch for the function of Exercise 11.6.2 that will cause the value it yields for $f(i)$ to be the same as that found for $f(i)$ in Example 11.6.4. Although Exercise 11.6.2 and Example 11.6.4 describe the same multivalued function, the specific values assigned for various z will not agree everywhere, due to the difference in the location of the branch cuts. Identify the portions of the complex plane where both these descriptions do and do not agree, and characterize the differences.

11.6.5 Find all singularities of

$$z^{-1/3} + \frac{z^{-1/4}}{(z-3)^3} + (z-2)^{1/2},$$

and identify their types (e.g., second-order branch point, fifth-order pole, ...). Include any singularities at the point at infinity.

Note. A branch point is of nth order if it requires n, but no fewer, circuits around the point to restore the original value.

11.6.6 The function $F(z) = \ln(z^2 + 1)$ is made single-valued by straight-line branch cuts from $(x, y) = (0, -1)$ to $(-\infty, -1)$ and from $(0, +1)$ to $(0, +\infty)$. See Fig. 11.17. If $F(0) = -2\pi i$, find the value of $F(i - 2)$.

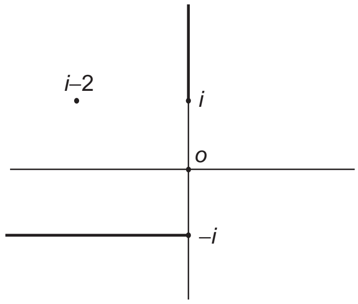

FIGURE 11.17 Branch cuts for Exercise 11.6.6.

11.6.7 Show that negative numbers have logarithms in the complex plane. In particular, find $\ln(-1)$.

ANS. $\ln(-1) = i\pi$.

11.6.8 For noninteger m, show that the binomial expansion of Exercise 11.5.2 holds only for a suitably defined branch of the function $(1+z)^m$. Show how the z-plane is cut. Explain why $|z| < 1$ may be taken as the circle of convergence for the expansion of this branch, in light of the cut you have chosen.

11.6.9 The Taylor expansion of Exercises 11.5.2 and 11.6.8 is **not** suitable for branches other than the one suitably defined branch of the function $(1+z)^m$ for noninteger m. (Note that other branches cannot have the same Taylor expansion since they must be distinguishable.) Using the same branch cut of the earlier exercises for all other branches, find the corresponding Taylor expansions, detailing the phase assignments and Taylor coefficients.

11.6.10 (a) Develop a Laurent expansion of $f(z) = [z(z-1)]^{-1}$ about the point $z = 1$ valid for small values of $|z-1|$. Specify the exact range over which your expansion holds. This is an analytic continuation of the infinite series in Eq. (11.49).

(b) Determine the Laurent expansion of $f(z)$ about $z = 1$ but for $|z-1|$ large.

Hint. Make a partial fraction decomposition of this function and use the geometric series.

11.6.11 (a) Given $f_1(z) = \int_0^\infty e^{-zt}\,dt$ (with t real), show that the domain in which $f_1(z)$ exists (and is analytic) is $\Re e(z) > 0$.

(b) Show that $f_2(z) = 1/z$ equals $f_1(z)$ over $\Re e(z) > 0$ and is therefore an analytic continuation of $f_1(z)$ over the entire z-plane except for $z = 0$.

(c) Expand $1/z$ about the point $z = -i$. You will have

$$f_3(z) = \sum_{n=0}^{\infty} a_n (z+i)^n.$$

What is the domain of this formula for $f_3(z)$?

ANS. $\displaystyle \frac{1}{z} = i \sum_{n=0}^{\infty} i^{-n}(z+i)^n, \quad |z+i| < 1.$