Example 11.5.1 LAURENT EXPANSION

Let $f(z) = [z(z-1)]^{-1}$. If we choose to make the Laurent expansion about $z_0 = 0$, then $r > 0$ and $R < 1$. These limitations arise because $f(z)$ diverges both at $z = 0$ and $z = 1$. A partial fraction expansion, followed by the binomial expansion of $(1-z)^{-1}$, yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots = -\sum_{n=-1}^{\infty} z^n. \tag{11.49}$$

From Eqs. (11.49), (11.47), and (11.48), we then have

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \ge -1, \\ 0 & \text{for } n < -1, \end{cases} \tag{11.50}$$

where the contour for Eq. (11.50) is counterclockwise in the annular region between $z' = 0$ and $|z'| = 1$.

The integrals in Eq. (11.50) can also be directly evaluated by insertion of the geometric-series expansion of $(1-z')^{-1}$:

$$a_n = \frac{-1}{2\pi i}\oint \sum_{m=0}^{\infty} (z')^m \frac{dz'}{(z')^{n+2}}. \tag{11.51}$$

Upon interchanging the order of summation and integration (permitted because the series is uniformly convergent), we have

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty} \oint (z')^{m-n-2}\, dz'. \tag{11.52}$$

The integral in Eq. (11.52) (including the initial factor $1/2\pi i$, but not the minus sign) was shown in Exercise 11.4.1 to be an integral representation of the Kronecker delta, and is therefore equal to $\delta_{m,n+1}$. The expression for a_n then reduces to

$$a_n = -\sum_{m=0}^{\infty} \delta_{m,n+1} = \begin{cases} -1, & n \ge -1, \\ 0, & n < -1, \end{cases}$$

in agreement with Eq. (11.50). ■

Exercises

11.5.1 Develop the Taylor expansion of $\ln(1+z)$.

ANS. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{z^n}{n}$.

11.5.2 Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2}z^2 + \cdots = \sum_{n=0}^{\infty}\binom{m}{n}z^n$$

for m, any real number. The expansion is convergent for $|z| < 1$. Why?

11.5.3 A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)| < 1$ for $|z| \le 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \le 1$.

Hint. One approach is to show that $f(z)/z$ is analytic and then to express $[f(z_0)/z_0]^n$ by the Cauchy integral formula. Finally, consider absolute magnitudes and take the nth root. This exercise is sometimes called Schwarz's theorem.

11.5.4 If $f(z)$ is a real function of the complex variable $z = x + iy$, that is, $f(x) = f^*(x)$, and the Laurent expansion about the origin, $f(z) = \sum a_n z^n$, has $a_n = 0$ for $n < -N$, show that all of the coefficients a_n are real.

Hint. Show that $z^N f(z)$ is analytic (via Morera's theorem, Section 11.4).

11.5.5 Prove that the Laurent expansion of a given function about a given point is unique; that is, if

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n = \sum_{n=-N}^{\infty} b_n (z - z_0)^n,$$

show that $a_n = b_n$ for all n.

Hint. Use the Cauchy integral formula.

11.5.6 Obtain the Laurent expansion of e^z/z^2 about $z = 0$.

11.5.7 Obtain the Laurent expansion of $ze^z/(z-1)$ about $z = 1$.

11.5.8 Obtain the Laurent expansion of $(z-1)\,e^{1/z}$ about $z = 0$.

11.6 SINGULARITIES

Poles

We define a point z_0 as an **isolated** singular point of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighboring points. There will therefore be a Laurent expansion about an isolated singular point, and one of the following statements will be true:

1. The most negative power of $z - z_0$ in the Laurent expansion of $f(z)$ about $z = z_0$ will be some finite power, $(z - z_0)^{-n}$, where n is an integer, or
2. The Laurent expansion of $f(z)$ about $z - z_0$ will continue to negatively infinite powers of $z - z_0$.