Exercises

Unless explicitly stated otherwise, closed contours occurring in these exercises are to be understood as traversed in the mathematically positive (counterclockwise) direction.

11.4.1 Show that

$$\frac{1}{2\pi i}\oint z^{m-n-1}dz, \quad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once), is a representation of the Kronecker δ_{mn}.

11.4.2 Evaluate

$$\oint_C \frac{dz}{z^2-1},$$

where C is the circle $|z-1|=1$.

11.4.3 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

11.4.4 You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Using mathematical induction (Section 1.4), prove that this expression is correct.

11.4.5 (a) A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z) \neq 0$ within C and $|f(z)| \le M$ on C, show that

$$|f(z)| \le M$$

for all points within C.

Hint. Consider $w(z) = 1/f(z)$.

(b) If $f(z) = 0$ within the contour C, show that the foregoing result does not hold and that it is possible to have $|f(z)| = 0$ at one or more points in the interior with $|f(z)| > 0$ over the entire bounding contour. Cite a specific example of an analytic function that behaves this way.

11.4.6 Evaluate

$$\oint_C \frac{e^{iz}}{z^3}dz,$$

for the contour a square with sides of length $a > 1$, centered at $z = 0$.

11.4.7 Evaluate

$$\oint_C \frac{\sin^2 z - z^2}{(z-a)^3}\,dz,$$

where the contour encircles the point $z = a$.

11.4.8 Evaluate

$$\oint_C \frac{dz}{z(2z+1)},$$

for the contour the unit circle.

11.4.9 Evaluate

$$\oint_C \frac{f(z)}{z(2z+1)^2}\,dz,$$

for the contour the unit circle.

Hint. Make a partial fraction expansion.

11.5 LAURENT EXPANSION

Taylor Expansion

The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series (Section 1.2), but this time for functions of a complex variable. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius less than $|z_1 - z_0|$ (Fig. 11.8). Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic, $f(z)$ is necessarily analytic on and within C.

From the Cauchy integral formula, Eq. (11.30),

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}. \end{aligned} \qquad (11.38)$$

Here z' is a point on the contour C and z is any point interior to C. It is not legal yet to expand the denominator of the integrand in Eq. (11.38) by the binomial theorem, for