## *Exercises*

**11.3.1** Show that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz.$

**11.3.2** Prove that $\left|\int_C f(z)dz\right| \le |f|_{\max} \cdot L,$

where $|f|_{\max}$ is the maximum value of $|f(z)|$ along the contour $C$ and $L$ is the length of the contour.

**11.3.3** Show that the integral

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)\,dz$$

has the same value on the two paths: (a) the straight line connecting the integration limits, and (b) an arc on the circle $|z| = 5$.

**11.3.4** Let $F(z) = \int_{\pi(1+i)}^{z} \cos 2\zeta\, d\zeta.$

Show that $F(z)$ is independent of the path connecting the limits of integration, and evaluate $F(\pi i)$.

**11.3.5** Evaluate $\oint_C (x^2 - iy^2)\,dz$, where the integration is (a) clockwise around the unit circle, (b) on a square with vertices at $\pm 1 \pm i$. Explain why the results of parts (a) and (b) are or are not identical.

**11.3.6** Verify that

$$\int_0^{1+i} z^* dz$$

depends on the path by evaluating the integral for the two paths shown in Fig. 11.7. Recall that $f(z) = z^*$ is not an analytic function of $z$ and that Cauchy's integral theorem therefore does not apply.

**11.3.7** Show that

$$\oint_C \frac{dz}{z^2 + z} = 0,$$

in which the contour $C$ is a circle defined by $|z| = R > 1$.

*Hint*. Direct use of the Cauchy integral theorem is illegal. The integral may be evaluated by expanding into partial fractions and then treating the two terms individually. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

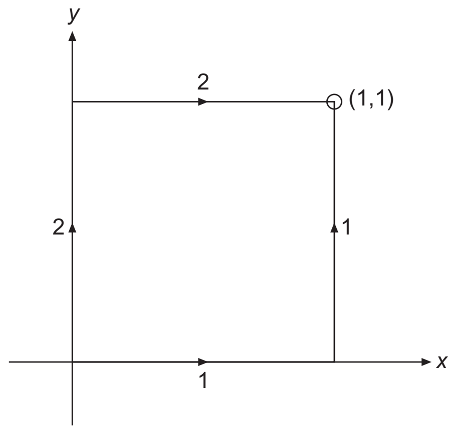


FIGURE 11.7 Contours for Exercise 11.3.6.

## 11.4 CAUCHY'S INTEGRAL FORMULA

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour $C$ and within the interior region bounded by $C$. This means that the contour $C$ is to be traversed in the **counterclockwise** direction. We seek to prove the following result, known as **Cauchy's integral formula**:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0), \tag{11.30}$$

in which $z_0$ is any point in the interior region bounded by $C$. Note that since $z$ is on the contour $C$ while $z_0$ is in the interior, $z - z_0 \neq 0$ and the integral Eq. (11.30) is well defined. Although $f(z)$ is assumed analytic, the integrand is $f(z)/(z-z_0)$ and is not analytic at $z = z_0$ unless $f(z_0) = 0$. We now deform the contour, to make it a circle of small radius $r$ about $z = z_0$, traversed, like the original contour, in the counterclockwise direction. As shown in the preceding section, this does not change the value of the integral. We therefore write $z = z_0 + re^{i\theta}$, so $dz = ire^{i\theta}d\theta$, the integration is from $\theta = 0$ to $\theta = 2\pi$, and

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta.$$

Taking the limit $r \to 0$, we obtain

$$\oint_C \frac{f(z)}{z-z_0}dz = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0), \tag{11.31}$$

where we have replaced $f(z)$ by its limit $f(z_0)$ because it is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an arbitrary interior point $z = z_0$ once the values on the boundary $C$ are specified.