Exercises

11.2.1 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

11.2.2 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that neither $u(x, y)$ nor $v(x, y)$ **can have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have saddle points only.)

11.2.3 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

(a) if $u(x, y) = x^3 - 3xy^2$, (b) if $v(x, y) = e^{-y} \sin x$.

11.2.4 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

11.2.5 Starting from $f(z) = 1/(x + iy)$, show that $1/z$ is analytic in the entire finite z plane except at the point $z = 0$. This extends our discussion of the analyticity of z^n to negative integer powers n.

11.2.6 Show that given the Cauchy-Riemann equations, the derivative $f'(z)$ has the same value for $dz = a\,dx + ib\,dy$ (with neither a nor b zero) as it has for $dz = dx$.

11.2.7 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Theta(r,\theta)}$, in which $R(r, \theta)$ and $\Theta(r, \theta)$ are differentiable real functions of r and θ, show that the Cauchy-Riemann conditions in polar coordinates become

$$\text{(a)} \quad \frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}, \qquad \text{(b)} \quad \frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}.$$

Hint. Set up the derivative first with δz radial and then with δz tangential.

11.2.8 As an extension of Exercise 11.2.7 show that $\Theta(r, \theta)$ satisfies the 2-D Laplace equation in polar coordinates,

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0.$$

11.2.9 For each of the following functions $f(z)$, find $f'(z)$ and identify the maximal region within which $f(z)$ is analytic.

(a) $f(z) = \dfrac{\sin z}{z}$,

(b) $f(z) = \dfrac{1}{z^2 + 1}$,

(c) $f(z) = \dfrac{1}{z(z + 1)}$,

(d) $f(z) = e^{-1/z}$,

(e) $f(z) = z^2 - 3z + 2$,

(f) $f(z) = \tan(z)$,

(g) $f(z) = \tanh(z)$.

11.2.10 For what complex values do each of the following functions $f(z)$ have a derivative?

(a) $f(z) = z^{3/2}$,

(b) $f(z) = z^{-3/2}$,

(c) $f(z) = \tan^{-1}(z)$,

(d) $f(z) = \tanh^{-1}(z)$.

11.2.11 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:

(a) Show that $df/dz = V_x - iV_y$.

(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks).

(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

11.2.12 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ with respect to z^* does not exist unless $f(z)$ is a constant.

Hint. Use the chain rule and take $x = (z + z^*)/2$, $y = (z - z^*)/2i$.

Note. This result emphasizes that our analytic function $f(z)$ is not just a complex function of two real variables x and y. It is a function of the complex variable $x + iy$.

11.3 Cauchy's Integral Theorem

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a path in the complex plane (known as a **contour**) may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour, from z_0 to z_0', designated C, into n intervals by picking $n-1$ intermediate points $z_1, z_2, \ldots$ on the contour (Fig. 11.2). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If $\lim_{n\to\infty} S_n$ exists, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\,dz = \int_C f(z)\,dz. \tag{11.16}$$

The right-hand side of Eq. (11.16) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$).