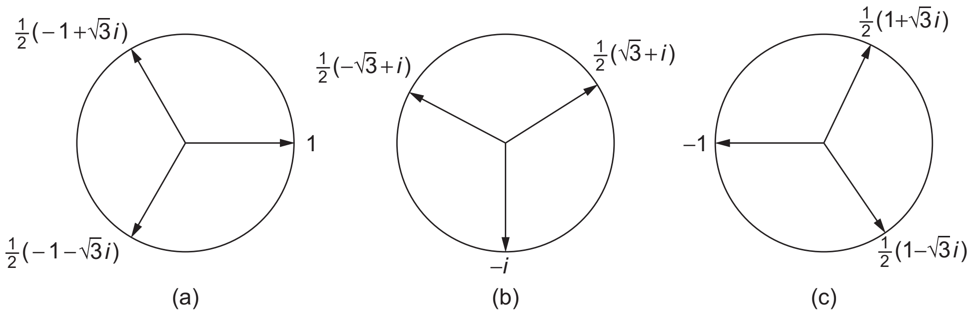

FIGURE 1.15 Cube roots: (a) $1^{1/3}$; (b) $i^{1/3}$; (c) $(-1)^{1/3}$.

Logarithm

Another multivalued complex function is the logarithm, which in the polar representation takes the form

$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta.$$

However, it is also true that

$$\ln z = \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i(\theta + 2n\pi), \tag{1.138}$$

for **any** positive or negative integer n. Thus, $\ln z$ has, for a given z, the infinite number of values corresponding to all possible choices of n in Eq. (1.138).

Exercises

1.8.1 Find the reciprocal of $x + iy$, working in polar form but expressing the final result in Cartesian form.

1.8.2 Show that complex numbers have square roots and that the square roots are contained in the complex plane. What are the square roots of i?

1.8.3 Show that

(a) $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots,$

(b) $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots.$

1.8.4 Prove that

(a) $\displaystyle\sum_{n=0}^{N-1}\cos nx = \frac{\sin(Nx/2)}{\sin x/2}\cos(N-1)\frac{x}{2},$

(b) $\displaystyle\sum_{n=0}^{N-1}\sin nx = \frac{\sin(Nx/2)}{\sin x/2}\sin(N-1)\frac{x}{2}.$

These series occur in the analysis of the multiple-slit diffraction pattern.

1.8.5 Assume that the trigonometric functions and the hyperbolic functions are defined for complex argument by the appropriate power series. Show that

$$i \sin z = \sinh iz, \ \sin iz = i \sinh z,$$

$$\cos z = \cosh iz, \ \cos iz = \cosh z.$$

1.8.6 Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

established from comparison of power series, show that

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$

$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$

This demonstrates that we may have $|\sin z|, |\cos z| > 1$ in the complex plane.

1.8.7 From the identities in Exercises 1.8.5 and 1.8.6 show that

(a) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$

$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$

(b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y, \quad |\cosh z|^2 = \cosh^2 x + \sin^2 y.$

1.8.8 Show that

(a) $\tanh \frac{z}{2} = \frac{\sinh x + i \sin y}{\cosh x + \cos y},$ (b) $\coth \frac{z}{2} = \frac{\sinh x - i \sin y}{\cosh x - \cos y}.$

1.8.9 By comparing series expansions, show that $\tan^{-1} x = \frac{i}{2} \ln \left(\frac{1 - ix}{1 + ix} \right).$

1.8.10 Find the Cartesian form for **all values** of

(a) $(-8)^{1/3},$

(b) $i^{1/4},$

(c) $e^{i\pi/4}.$

1.8.11 Find the polar form for **all values** of

(a) $(1 + i)^3,$

(b) $(-1)^{1/5}.$