III. STRESS-STRAIN RELATIONS. GENERALIZED HOOKE'S LAW

3.1 DIRECT FORM. INVERSE FORM.

$$\epsilon_{ij} = A_{ijkl}\sigma_{kl}, \qquad i,j,k,l = 1,2,3 \tag{3.1-1}$$

where

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij} \tag{3.1-2}$$

e.g.

$$\begin{aligned} \epsilon_{11} = {} & A_{1111}\sigma_{11} + A_{1112}\sigma_{12} + A_{1113}\sigma_{13} \\ & + A_{1121}\sigma_{21} + A_{1122}\sigma_{22} + A_{1123}\sigma_{23} \\ & + A_{1131}\sigma_{31} + A_{1132}\sigma_{32} + A_{1133}\sigma_{33} \end{aligned} \tag{3.1-3}$$

or alternatively,

$$\underset{6\times1}{\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{Bmatrix}} = \underset{6\times6}{\begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1113} & A_{1123} \\ A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2213} & A_{2223} \\ A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3313} & A_{3323} \\ A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1213} & A_{1223} \\ A_{1311} & A_{1322} & A_{1333} & A_{1312} & A_{1213} & A_{1323} \\ A_{2311} & A_{2322} & A_{2333} & A_{1312} & A_{2313} & A_{2323} \end{bmatrix}} \underset{6\times1}{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{Bmatrix}} \tag{3.1-4}$$

Inverse Form

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}, \qquad i,j,k,l = 1,2,3 \tag{3.1-5}$$

where

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{3.1-6}$$

e.g.

$$\begin{aligned}\sigma_{12} = C_{1211}\epsilon_{11} + C_{1212}\epsilon_{12} + C_{1213}\epsilon_{13} \\ + C_{1221}\epsilon_{21} + C_{1222}\epsilon_{22} + C_{1223}\epsilon_{23} \\ + C_{1232}\epsilon_{31} + C_{1232}\epsilon_{32} + C_{1233}\epsilon_{33}\end{aligned} \tag{3.1-7}$$

or alternatively,

$$\underset{6\text{x}1}{\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{Bmatrix}} = \underset{6\text{x}6}{\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123}\\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223}\\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323}\\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223}\\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323}\\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323}\end{bmatrix}} \underset{6\text{x}1}{\begin{Bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{12}\\ \epsilon_{13}\\ \epsilon_{23}\end{Bmatrix}} \tag{3.1-8}$$

3.2 SPECIAL SYMMETRY

Upon coordinate transformation, Fig.A.1-2,

$$A'_{ijkl} = a_{ip}a_{jq}a_{kr}a_{\ell s}A_{pqrs} \tag{3.2-1}$$

where a_{ip}, etc., are direction cosines between the $\bar{x}'$ and the $\bar{x}$ co-ordinates.

<u>Symmetry With Respect to One Plane</u>: monoclinic material, Fig.3.2-1

a_{ij}	x_1'	x_2'	x_3'
x_1	1	0	0
x_2	0	1	0
x_3	0	0	-1

The stiffness matrix becomes:

$$\begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & 0 & 0 \\ A_{1122} & A_{2222} & A_{2233} & A_{2212} & 0 & 0 \\ A_{1133} & A_{2233} & A_{3333} & A_{3312} & 0 & 0 \\ A_{1112} & A_{2212} & A_{3312} & A_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{1313} & A_{1323} \\ 0 & 0 & 0 & 0 & A_{1323} & A_{2323} \end{bmatrix} \qquad (3.2\text{-}2)$$

The moduli matrix C_{ijkl} are similar.

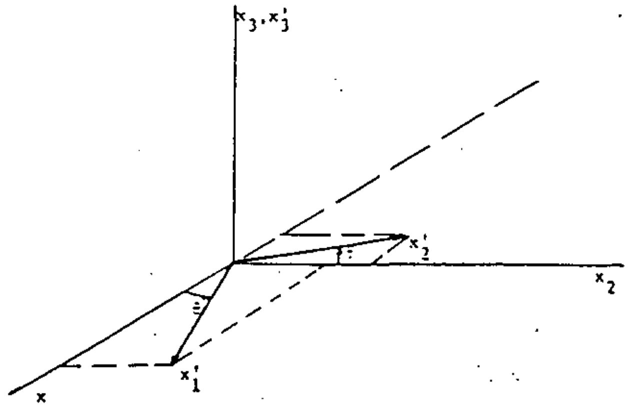

Fig.3.2-3 Transversely isotropic system

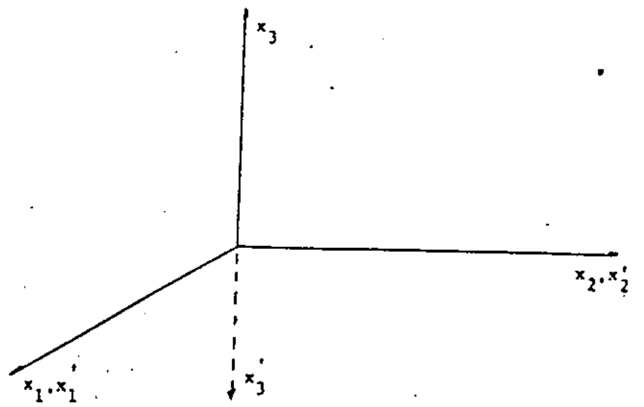

Fig.3.2-1 Monodinic system

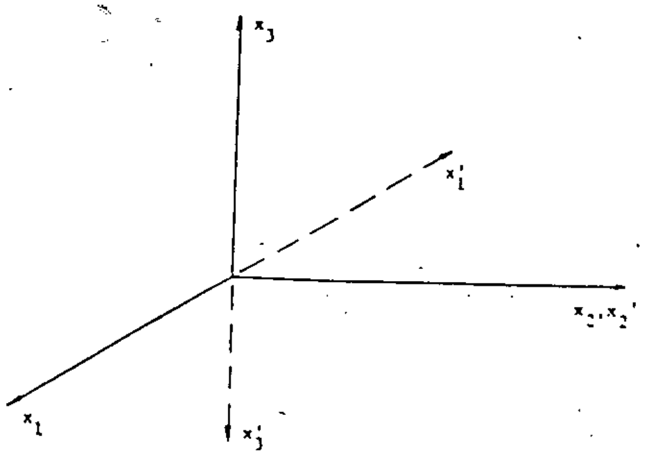

Fig.3.2-2 Orthotropic system

Symmetry With Respect to Two Orthogonal Planes: orthotropic material, Fig.3.2-2,

a_{ij}	x_1'	x_2'	x_3'
x_1	-1	0	0
x_2	0	1	0
x_3	0	0	-1

The stiffness matrix becomes:

$$\begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & 0 & 0 & 0 \\ A_{1122} & A_{2222} & A_{2233} & 0 & 0 & 0 \\ A_{1133} & A_{2233} & A_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{2323} \end{bmatrix} \tag{3.2-3}$$

The moduli matrix C_{ijkl} is similar.

Symmetry With Respect to One Axis: transversely isotropic or cross anisotropic material, Fig.3.2-3,

a_{ij}	x_1'	x_2'	x_3'
x_1	$\cos\theta$	$-\sin\theta$	0
x_2	$\sin\theta$	$\cos\theta$	0
x_3	0	0	1

The stiffness matrix is:

$$\begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & 0 & 0 & 0 \\ A_{1122} & A_{1111} & A_{1133} & 0 & 0 & 0 \\ A_{1133} & A_{1133} & A_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(A_{1111}-A_{1122}) & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{1313} & 0 \\ 0 & 0 & 0 & 0 & & A_{1313} \end{bmatrix} \qquad (3.2\text{-}4)$$

The compliance matrix is similar in form.

<u>Rotational Symmetry With Respect to Two Mutually Perpendicular Axes</u>:

isotropic material

The stiffness matrix is:

$$\begin{bmatrix} A_{1111} & A_{1122} & A_{1122} & 0 & 0 & 0 \\ A_{1122} & A_{1111} & A_{1122} & & 0 & 0 \\ A_{1122} & A_{1122} & A_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(A_{1111}-A_{1122}) & 0 & \\ 0 & 0 & 0 & 0 & \frac{1}{2}(A_{1111}-A_{1122}) & \\ C & 0 & 0 & 0 & 0 & \frac{1}{2}(A_{1111}-A_{1122}) \end{bmatrix}$$

(3.2-5)

The compliance matrix takes a form that is similar to the above.

3.3 CONVENTIONAL ELASTIC CONSTANTS FOR ISOTROPIC MATERIAL.

λ, μ	Lame' constants
E	Young's modulus
G	shear modulus, $G = \mu$
K	bulk modulus
ν	Poisson's ratio

$$A_{1122} = \lambda, \; A_{1212} = \tfrac{1}{2}(A_{1111} - A_{1122}), \; A_{1111} = \lambda + 2\mu \qquad (3.3\text{-}1)$$

$$\begin{aligned} C_{1111} &= (\lambda + \mu)/\mu\,(3\lambda + 2\mu) \\ C_{1122} &= -\lambda/\,2\mu\,(3\lambda + 2\mu) \\ \tfrac{1}{2}(C_{1111} - C_{1122}) &= 1/4\mu \end{aligned} \qquad (3.3\text{-}2)$$

3.4 STRESS-STRAIN RELATIONS. ISOTROPIC MATERIALS

$$\epsilon_{ij} = \sigma_{ij}/2\mu - [\lambda/2\mu(3\lambda+2\mu)]\,\sigma_{kk}\,\delta_{ij} \qquad (3.4\text{-}1)$$

$$\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\,\delta_{ij} \qquad (3.4\text{-}2)$$

<u>Rectangular Co-ordinates (x,y,z)</u>

Direct form: $\sigma = [\sigma_{xx} + \sigma_{yy} + \sigma_{zz}]/3$

$$2\mu\epsilon_{xx} = \sigma_{xx} - 3[\nu/(1+\nu)]\,\sigma \qquad (3.4\text{-}3a)$$

$$2\mu\epsilon_{yy} = \sigma_{yy} - 3[\nu/(1+\nu)]\,\sigma \qquad (3.4\text{-}3b)$$

$$2\mu\epsilon_{zz} = \sigma_{zz} - 3[\nu/(1+\nu)]\,\sigma \qquad (3.4\text{-}3c)$$

$$2\mu\epsilon_{xy} = \sigma_{xy} \qquad (3.4\text{-}3d)$$

$$2\mu\epsilon_{yz} = \sigma_{yz} \qquad (3.4\text{-}3e)$$

$$2\mu\epsilon_{zx} = \sigma_{zx} \qquad (3.4\text{-}3f)$$

where

$$\sigma = [\sigma_{xx} + \sigma_{yy} + \sigma_{zz}]/3 \qquad (3.4\text{-}4)$$

3.5 PROBLEMS

1. An elastic layer, of moduli E and ν, is enclosed in a rigid container with one end open. The layer is then compressed by a rigid stamp which applies a direct stress σ_z to the layer. Let the apparent Young's modulus be E'(that is, σ_z/ϵ_z). Plot the ratio E'/E as a function of ν, knowing that $0 \le \nu \le 0.5$.

2. Prove that Eq.(3.5-1) follows from Eqs.(3.5-2),(3.5-3)

$$e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = [(1-2\nu)/E](\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \qquad (3.5\text{-}1)$$

$$\left.\begin{aligned} \sigma_{xx} &= \lambda e + 2\mu\epsilon_{xx} \\ \sigma_{yy} &= \lambda e + 2\mu\epsilon_{yy} \\ \sigma_{zz} &= \lambda e + 2\mu\epsilon_{zz} \end{aligned}\right\} \qquad (3.5\text{-}2)$$

$$\mu = E/[2(1+\nu)], \qquad \lambda = \nu E/[(1+\nu)(1-2\nu)] \qquad (3.5\text{-}3)$$

3. Show that the assumption that an elastic isotropic solid is incompressible is equivalent to the assumption that Poisson's ratio is equal to one half.

*4. Prove that the principal axes of stress coincide with the principal axes of strain for a linear, isotropic, elastic solid, and relate the principal values to each other.

5. Show that the multiaxial isotropic Hooke's law can be obtained by the following argument: An elememtary rectangular parallelepiped subjected to tensile stresses σ_{xx} on opposite faces will experience a longitudinal extension $\epsilon_{xx} = \sigma_{xx}/E$ and lateral contractions $\epsilon_{yy} = \epsilon_{zz} = -\nu\epsilon_{xx}$. Now consider the effect of stresses σ_{xx}, σ_{yy}, σ_{zz} and superpose the resulting strains.

6. Show that the index notation of the isotropic Hooke's law may be written as

$$\sigma_{ij} = 2\mu[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}] \qquad (3.5\text{-}4)$$

7. Show directly from the generalized Hooke's law that in an isotropic body the principal axes of strain coincide with those of stress. Hint: Take the coordinate axes along the principal axes of strain and consider the effect on σ_{23} and σ_{31} of a rotation of axes by 180^{o} about the x_3-axis.

8. For an isotropic linear elastic solid, derive the following relations between the Lamé constants λ and μ, Poisson's ratio ν, Young's modulus E, and the bulk modulus K:

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\nu(E-2\mu)}{3\mu-E} = K - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$= \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E}$$

$$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)}$$

$$= \frac{3KE}{9K-E},$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3K-\lambda} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)}$$

$$= \frac{3K-E}{6K},$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda}$$

$$= 2\mu(1+\nu) = \frac{9K\mu}{3K+\mu} = 3K(1-2\nu),$$

$$K = \lambda + \frac{2}{3}\mu = (\lambda+E)/2 + \sqrt{(E-3\lambda)^2 + 8\lambda E}/6$$

$$= \lambda(1+\nu)/(3\nu) = \mu E/[3(3\mu-E)]$$

$$= 2\mu(1+\nu)/[3(1-2\nu)] = E/[3(1-2\nu)]$$

TABLE 3.3-1 INTERRELATION BETWEEN THE ELASTIC CONSTANTS OF ISOTROPIC HOMOGENEOUS BODY

	λ	μ	E	ν	K
λ,μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ,E		$\frac{(E-3\lambda)+A^{*}}{4}$		$\frac{-(E+\lambda)+A^{*}}{4\lambda}$	$\frac{3(\lambda+E)+A^{*}}{6}$
λ,ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ,K		$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	
μ,E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ,ν	$\frac{2\mu\nu}{1-2\nu}$		$2\nu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ,K	$\frac{3K-2\mu}{3}$		$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	
E,ν	$\frac{\nu E}{(1-2\nu)(1+\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
EK	$\frac{3K(3)-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν,K	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

$$A^{*} = \sqrt{(E-3\lambda)^{2}+8\lambda E}$$

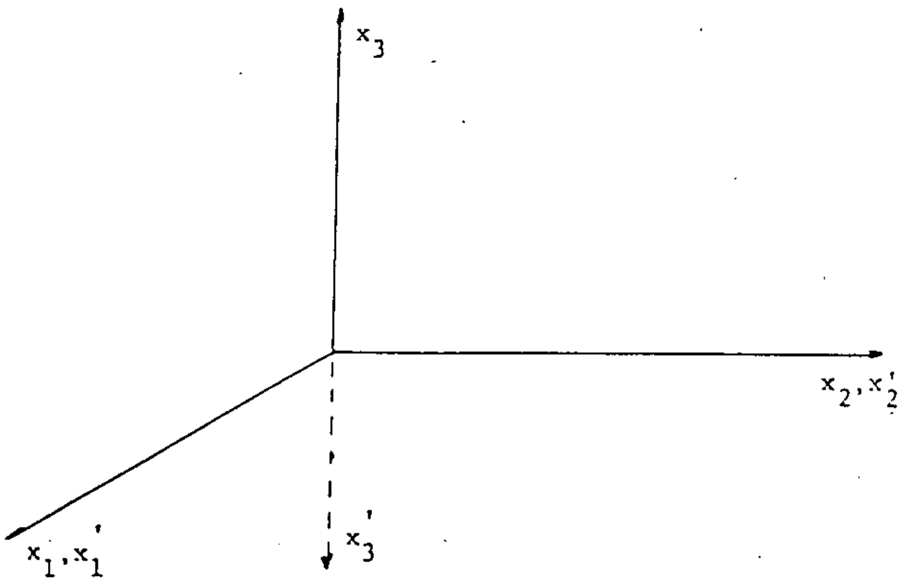

Fig.3.2-1 Monodinic system

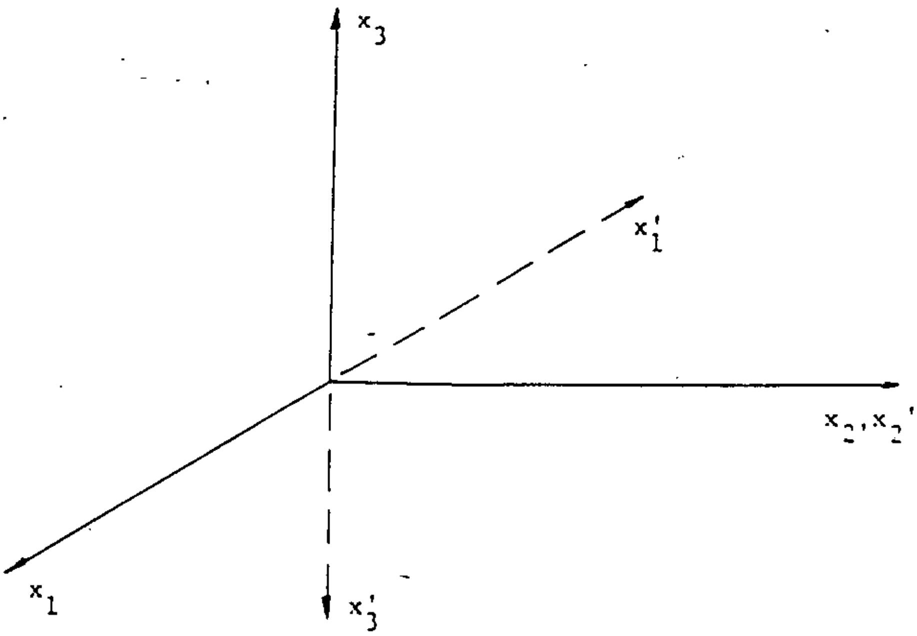

Fig.3.2-2 Orthotropic system

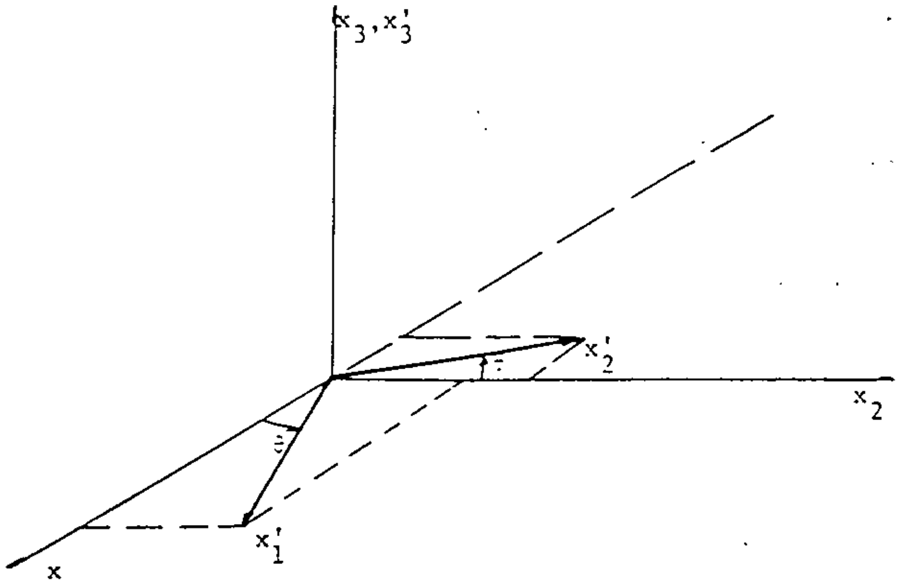

Fig.3.2-3 Transversely isotropic system