II. THEORY OF STRAIN

2.1 DISPLACEMENT. DEFORMATION.

When a material is subjected to applied surface forces, body forces, temperature change, moisture change or other environmental changes, each of its material particles or materials points is caused to occupied a new position, see Fig.2.1-1. The position vector from P to P' is called the displacement of the material point P. The material body or the solid is said to be mapped onto its new configuration. The displacement consists of two parts: the rigid body displacement and deformation. The deformation is the part of displacement that contributes to changes of shape and size of a solid. The changes are measured in terms of shear strains and normal strains, respectively. Consider a triangle formed by the point P and two of its neighboring points, say Q and R. When mapped into the new configuration, P'Q'R', its size and shape may both change. The triangle PQR is said to be deformed. When there is no change in size or shape, the body is under at most a rigid body motion.

Rigid Body Translation

$$\bar{u} = \bar{d} \quad \text{or} \quad u_i = d_i = (d_x, d_y, d_z) \qquad (2.1\text{-}1)$$

where $\bar{u}$ is the displacement and (d_x, d_y, d_z) are constants.

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Rigid Body Rotation

Consider rigid body rotation about the z-axis, Fig.2.1-2. It is clear that

$$d_x' = \cos\theta \quad d_x - \sin\theta \quad d_y \tag{2.1-2a}$$

$$d_y' = \sin\theta \quad d_x + \cos\theta \quad d_y \tag{2.1-2b}$$

$$d_z' = d_z \tag{2.1-2c}$$

where d_i and d_i' are displacements $\bar{OP}$ and $\bar{OP}'$, respectively, and the rotation tensor is given by:

$$[\bar{\omega}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1-3}$$

In general the rotation tensor is related to the rotation vector by

$$\bar{\omega} = \bar{\nabla} \times \bar{\omega}, \quad \text{or} \quad \omega_k = \epsilon_{ijk}\omega_{ij} \tag{2.1-4}$$

Deformation

Consider the displacement of two neighboring points P and Q, Fig.2.1-3. If there occured only rigid body displacement, the points P and Q would have the same displacement $\bar{u}$ and would lead to vanshing $d\bar{u}$. Deformation is therefore measured by the vector $d\bar{u}$ and its presence causes changes in lengths and angles in the neighborhood of point P. The small change in displacement in the neighborhood of point P can be written as:

$$du_x = (\partial u_x/\partial x)dx + (\partial u_x/\partial y)dy + (\partial u_x/\partial z)dz \qquad (2.1\text{-}5a)$$

$$du_y = (\partial u_y/\partial x)dx + (\partial u_y/\partial y)dy + (\partial u_y/\partial z)dz \qquad (2.1\text{-}5b)$$

$$du_z = (\partial u_z/\partial x)dx + (\partial u_z/\partial y)dy + (\partial u_z/\partial z)dz \qquad (2.1\text{-}5c)$$

or $$d\bar{u} = (\bar{\nabla}\bar{u}) \cdot d\bar{x}, \qquad du_i = (\partial u_i/\partial x_j)dx_j \qquad (2.1\text{-}5)$$

2.2 STATE OF STRAIN AT A POINT. SMALL STRAIN.

strains are measurements of changes in size or shape. Adopting the engineering definition of strain, the change of length per unit original length is called a normal strain and the change of angle from an initial right angle is called a shear strain. They are denoted by ϵ and γ, respectively. The normal strain can be defined as an average over a finite original length or defined as an infinitesimal quantity when the original length approaches zero.

Consider the changes that occured in the neighborhood of a point P, Fig.2.2-1. Assuming the angles α and β are small, it is clearly seen that

$$\epsilon_{xx} = \lim_{|\ell\Gamma| \to 0} [|P'Q'| - |PQ|]/|PQ| = \partial u_x/\partial x \qquad (2.2\text{-}1a)$$

$$\epsilon_{yy} = \lim_{|\ell\theta| \to 0} [|P'R'| - |PR|]/|PR| = \partial u_y/\partial y \qquad (2.2\text{-}1b)$$

$$\gamma_{xy} = \alpha + \beta = \angle R'P'Q' - \angle RPQ = \partial u_y/\partial x + \partial u_x/\partial y \qquad (2.2\text{-}1c)$$

where $\alpha \cong \tan\alpha$, and $\beta \cong \tan\beta$ and the subscripts are used to indicate the directions of measurements for deformation, original length, and those of the lines that form an angle. It is noted that γ_{xy} is symmetric in x and in y. This implies that a rectangle always deforms into a rhombus.

When the third dimension is included, i.e. the z-dimension, three additional measurements can be made in the neighborhood of point P. These are

$$\epsilon_{zz} = \partial u_z/\partial z \qquad (2.2\text{-}1d)$$

$$\gamma_{zx} = \gamma_{xz} = \partial u_z/\partial x + \partial u_x/\partial z \qquad (2.2\text{-}1e)$$

$$\gamma_{zy} = \gamma_{yz} = \partial u_y/\partial z + \partial u_z/\partial y \qquad (2.2\text{-}1f)$$

The state of strain at a point P or the charaterization of the changes in size and in shape is given by the strain tensor $\tilde{\epsilon}$ where

$$[\tilde{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

where $\epsilon_{xy} = \gamma_{xy}/2$, etc., are the averages of angle changes.

It is noted that the components of strain are consisted of the symmetric part of the gradient of the displacement. Consider again Equation (2.1-5) and decompose it into symmetric and skew-symmetric parts, it is seen that

$$du_i = \epsilon_{ij} dx_j + \omega_{ij} dx_j,$$

where

$$\epsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}], \quad \omega_{ij} = \frac{1}{2}[u_{i,j} - u_{j,i}]$$

where ϵ_{ij} and ω_{ij} are the strain tensor and the <u>rotation tensor</u>, respectively.

*2.3 LARGE STRAIN CONSIDERATION

Consider the situation in Fig.2.1-3 again. It is clear that

$$\bar{e}^{*}\, ds^{*} = \bar{e}\, ds + d\bar{u} \tag{2.3-1}$$

where $\bar{e}^{*}$ and $\bar{e}$ are unit vectors along P'Q" and PQ, respectively. Simple manipulation leads to

$$ds^{*2} = d\bar{s}^{*} \,.\, d\bar{s}^{*} = [1 + \bar{e}\,.(\bar{\nabla}\bar{u}.\bar{e}) + \bar{e}.(\bar{\nabla}\bar{u}).(\bar{\nabla}\bar{u}).\bar{e}]\; ds^{2} \tag{2.3-2}$$

or

$$ds^{*2} = [1 + (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})e_i e_j]\; ds^{2} \tag{2.3-3}$$

The definition of strain leads to the follows:

$$\epsilon = (ds^{*} - ds)/ds = ds^{*}/ds - 1 \tag{2.3-4a}$$

$$\epsilon = [1 + (2\epsilon_{ij} + u_{k,i}u_{k,j})e_i e_j]^{1/2} - 1 \tag{2.3-4b}$$

<u>Example</u>

Let PQ be in the x-direction, or $\epsilon_i = (1,0,0)$, then

$$\epsilon = \epsilon_{xx} = [1+2\partial u_x/\partial x+(\partial u_x/\partial x)^2+(\partial u_y/\partial x)^2+(\partial u_z/\partial x)^2]^{1/2}-1 \tag{2.3-5}$$

If the gradiants are much smaller then 1, i.e. $\partial\nabla u/\partial x \ll 1$, then the square terms can be droped and the following is obtained

$$\epsilon_{xx} \approx [1 + 2\partial u_x/\partial x]^{1/2} - 1$$
$$\approx 1 + \frac{1}{2} \cdot 2\, \partial u_x/\partial x - 1$$
$$= \partial u_x/\partial x$$

2.4 DISPLACEMENT AND STATE OF STRAIN AT A POINT. NOTATION.

Rectangular Co-ordinates (x,y,z)

displacement vector $\bar{u}$: $\bar{u} = u_x\bar{e}_x + u_y\bar{e}_y + u_z\bar{e}_z$ (2.4-1)

strain tensor $\bar{\epsilon}$:

$$[\bar{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \quad (2.4\text{-}2)$$

Cylindrical Co-ordinates, (r,θ,z)

displacement vector $\bar{u}$: $\bar{u} = u_r\bar{e}_r + u_\theta\bar{e}_\theta + u_z\bar{e}_z$ (2.4-3)

strain tensor $\bar{\epsilon}$

$$[\bar{\epsilon}] = \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{bmatrix} \quad (2.4\text{-}4)$$

Spherical Co-ordinates, (R,θ,ϕ)

displacement vector $\bar{u}$: $\bar{u} = u_R\bar{e}_R + u_\theta\bar{e}_\theta + u_\phi\bar{e}_\phi$ (2.4-5)

strain tensor $\bar{\epsilon}$:

$$[\bar{\epsilon}] = \begin{bmatrix} \epsilon_{RR} & \epsilon_{R\theta} & \epsilon_{R\phi} \\ \epsilon_{\theta R} & \epsilon_{\theta\theta} & \epsilon_{\theta\phi} \\ \epsilon_{\phi R} & \epsilon_{\phi\theta} & \epsilon_{\phi\phi} \end{bmatrix} \qquad (2.4\text{-}6)$$

The gradient of a displacement vector is consisted of two parts:

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad (2.4\text{-}1)$$

where a comma denotes partial differentiation with respect to spatial co-ordinate(s) that follow(s) it and the symmetric part is called strain tensor and the anti-symmetric part is call the rotation tensor:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) = \epsilon_{ji} \qquad (2.4\text{-}2)$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad = -\omega_{ji} \qquad (2.4\text{-}3)$$

There is a rotation vector ω_k which is associated with the rotation tensor ω_{ij} and is defined as:

$$\omega_k = -\frac{1}{2} e_{kij}\omega_{ij} \qquad (2.4\text{-}4)$$

2.5 STRAIN-DISPLACEMENT RELATIONS

$$\bar{\epsilon} = \frac{1}{2}\bar{e}\cdot[(\bar{\nabla}\bar{u} + \bar{u}\bar{\nabla})\cdot\bar{e} + (\bar{\nabla}\bar{u})(\bar{\nabla}\bar{u})\cdot\bar{e}] \qquad (2.5\text{-}1a)$$

$$\epsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}] \qquad (2.5\text{-}1b)$$

Rectangular Co-ordinates, (x,y,z):

$$\epsilon_{xx} = u_{x,x} + \frac{1}{2}\left[(u_{x,x})^2 + (u_{y,x})^2 + (u_{z,x})^2\right] \quad (2.5\text{-}2a)$$

$$\epsilon_{yy} = u_{y,y} + \frac{1}{2}\left[(u_{x,y})^2 + (u_{y,y})^2 + (u_{z,y})^2\right] \quad (2.5\text{-}2b)$$

$$\epsilon_{zz} = u_{z,z} + \frac{1}{2}\left[(u_{x,z})^2 + (u_{y,z})^2 + (u_{z,z})^2\right] \quad (2.5\text{-}2c)$$

$$\epsilon_{xy} = \frac{1}{2}\left[u_{x,y} + u_{y,x} + u_{x,x}u_{x,y} + u_{y,x}u_{y,y} + u_{z,x}u_{z,y}\right] \quad (2.5\text{-}2d)$$

$$\epsilon_{yz} = \frac{1}{2}\left[u_{y,z} + u_{z,y} + u_{x,y}u_{x,z} + u_{y,y}u_{y,z} + u_{z,y}u_{z,z}\right] \quad (2.5\text{-}2e)$$

$$\epsilon_{zx} = \frac{1}{2}\left[u_{z,x} + u_{x,z} + u_{x,x}u_{x,z} + u_{y,x}u_{y,z} + u_{z,x}u_{z,z}\right] \quad (2.5\text{-}2f)$$

Cylindrical Co-ordinates, (r,θ,z):

$$\epsilon_{rr} = u_{r,r} + \frac{1}{2}\left[(u_{r,r})^2 + (u_{\theta,r})^2 + (u_{z,r})^2\right] \quad (2.5\text{-}3a)$$

$$\epsilon_{\theta\theta} = u_{\theta,\theta}/r + u_r/r + \frac{1}{2}\left[(u_{\theta,\theta}/r + u_r/r)^2 + (u_{r,\theta}/r - u_\theta/r)^2 + (u_{z,\theta}/r)^2\right] \quad (2.5\text{-}3B)$$

$$\epsilon_{zz} = u_{r,z} + \frac{1}{2}\left[(u_{r,z})^2 + (u_{\theta,z})^2 + (u_{z,z})^2\right] \quad (2.5\text{-}3c)$$

$$\epsilon_{r\theta} = \frac{1}{2}\left[u_{r,\theta}/r + u_{\theta,r} - u_\theta/r + u_{r,r}(u_{r,\theta}/r - u_\theta/r) + u_{\theta,r}(u_{\theta,\theta}/r + u_r/r)^2 - u_{z,r}u_{z,\theta}/r\right] \quad (2.5\text{-}3D)$$

$$\epsilon_{\theta z} = \frac{1}{2} [u_{\theta,z} + u_{z,\theta}/r + u_{r,z}(u_{r,\theta} - u_\theta)/r$$

$$+ u_{\theta,z}(u_{\theta,\theta} + u_r)/r + u_{z,z}u_{z,\theta}/r] \quad (2.5\text{-}3E)$$

$$\epsilon_{zr} = \frac{1}{2} [u_{r,z} + u_{z,r} + u_{r,r}u_{r,z} + u_{\theta,r}u_{\theta,z} + u_{z,r}u_{z,z}] \quad (2.5\text{-}3f)$$

<u>Spherical Co-ordinates (R,θ,ϕ):</u>

$$\epsilon_{RR} = u_{R,R} + \frac{1}{2} [(u_{R,R})^2 + (u_{\phi,R})^2 + (u_{\theta,R})^2] \quad (2.5\text{-}4a)$$

$$\epsilon_{\theta\theta} = u_{\theta,\theta}/(R \sin\phi) + u_R/R + (u_\phi \cot\phi)/R$$

$$+[(u_{R,\theta}/R - u_\theta \sin\phi/R)^2 + (u_{\phi,\theta}/R - u_\theta \cos\phi/R)^2$$

$$+(u_{\theta,\theta}/R + u_R \sin\phi/R + u_\phi \cos\phi/R)^2]/ \sin^2\phi \quad (2.5\text{-}4b)$$

$$\epsilon_{\phi\phi} = u_{\phi,\phi}/R + u_R/R + \frac{1}{2} [(u_{R,\phi}/R - u_\phi/R)^2$$

$$+ (u_{\phi,\phi}/R + u_R/R)^2 + (u_{\theta,\phi}/R)^2] \quad (2.5\text{-}4c)$$

$$\epsilon_{R\theta} = \frac{1}{2} (u_{\theta,R} + u_{R,\theta}/(R \sin\phi) - u_\theta/R$$

$$+[(u_{R,R})(u_{R,\theta} - u_\theta \sin\phi) + (u_{\phi,R})(u_{\phi,\theta} - u_\theta \cos\phi)$$

$$+(u_{\theta,R})(u_{\theta,\theta} + u_R \sin\phi + u_\phi \cos\phi)]/(R \sin\phi)) \quad (2.5\text{-}4d)$$

$$\epsilon_{\theta\phi} = \frac{1}{2}\{ u_{\phi,\theta}/(R \sin\phi) - \cot\theta\, u_\theta/R + u_{\theta,\phi}/R$$

$$+ [(u_{R,\phi} - u_\phi)(u_{R,\theta} - \sin\phi\, u_\theta)$$

$$+ (u_{\phi,\phi} + u_R)(u_{\phi,\theta} - \cot\phi\, u_\theta)$$

$$+ (u_{\theta,\phi})(u_{\theta,\theta} + \sin\phi\, u_R + \cos\phi\, u_\phi)]/(R \sin\phi)\} \quad (2.5\text{-}4e)$$

$$\epsilon_{\phi R} = \frac{1}{2}\{u_{\phi,R} + [-u_\phi + u_{R,\phi} + u_{R,R}(u_{R,\phi} - u_\phi)$$

$$+ u_{\phi,R}(u_{\phi,\phi} + u_R) + u_{\theta,R}u_{\theta,\phi}]/R\} \quad (2.5\text{-}4f)$$

For the case of small strains, only the linear terms are retained in Eqs.(2.5-2) to (2.5-4)

2.6 PRINCIPAL STRAINS AND DIRECTIONS. STRAIN INVARIANTS.

The direction cosines of the principal direction $\bar{n}$ for the strain tensor is determined by:

$$(\epsilon_{xx} - \epsilon)\, n_x + \epsilon_{yx}n_y + \epsilon_{zx}n_z = 0 \quad (2.6\text{-}1)$$

$$\epsilon_{yx}n_x + (\epsilon_{yy} - \epsilon)n_y + \epsilon_{yz}n_z = 0 \quad (2.6\text{-}2)$$

$$\epsilon_{zx}n_x + \epsilon_{zy}n_y + (\epsilon_{zz} - \epsilon)n_z = 0 \quad (2.6\text{-}3)$$

For non-trivial solution, the determinant of coefficients vanish, i.e.,

$$\begin{vmatrix} \epsilon_{xx}-\epsilon & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy}-\epsilon & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}-\epsilon \end{vmatrix} = 0 \qquad (2.6\text{-}4)$$

or

$$\epsilon^3 - I_\epsilon \epsilon^2 + II_\epsilon \epsilon - III_\epsilon = 0 \qquad (2.6\text{-}5)$$

where I_ϵ, II_ϵ and III_ϵ are the strain invariants and are defined as:

$$I_\epsilon = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \epsilon_1 + \epsilon_2 + \epsilon_3 \qquad (2.6\text{-}6)$$

$$II_\epsilon = \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} - \epsilon_{xy}^2 - \epsilon_{yz}^2 - \epsilon_{zx}^2 \qquad (2.6\text{-}7)$$

$$III_\epsilon = \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} - \epsilon_{xx}\epsilon_{yz}^2 - \epsilon_{yy}\epsilon_{zx}^2 - \epsilon_{zz}\epsilon_{xy}^2 + 2\epsilon_{xy}\epsilon_{yz}\epsilon_{zx} \qquad (2.6\text{-}8)$$

where three roots of the equation are the principal strains $\epsilon_1, \epsilon_2, \epsilon_3$. The direction of each principal strain can be obtained by using Eqs. (2.6-1) to (2.6-3)

2.7 VOLUMETRIC STRAIN. STRAIN DEVIATOR.

Spherical and Deviatoric Components

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij} \qquad (2.7\text{-}1)$$

where

$$\epsilon = \epsilon_{kk}/3 = (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})/3 \qquad (2.7\text{-}2)$$

is the spherical component and e_{ij} is the deviatoric component of the strain tensor.

Volumetric Strain

Specific volume change for small strain e_v:

$$e_v = \epsilon_{kk} = I_\epsilon \tag{2.7-3}$$

*2.8 STRAIN TRANSFORMATION

The strain tensor is of second rank and transforms according to the rules of co-ordinate transformation as follows:

$$\epsilon'_{ij} = a_{ip} a_{jq} \epsilon_{pq} \tag{2.8-1}$$

and the expanded equations can be obtained by replacing $\bar{\sigma}$ by $\bar{\epsilon}$ in Eqs.(1.6-1) to (1.6-31)

2.9 COMPATIBILITY CONDITIONS

Along any closed contour the single-valuedness in displacement field requires that

$$\oint du_i = 0 \qquad i = 1,2,3 \tag{2.9-1}$$

or

$$\epsilon_{ij,k\ell} + \epsilon_{k\ell,ij} - \epsilon_{ik,j\ell} - \epsilon_{j\ell,ki} = 0 \tag{2.9-2}$$

Rectangular Co-ordinates, (x,y,z)

$$\partial^2 \epsilon_{xx}/\partial y^2 + \partial^2 \epsilon_{yy}/\partial x^2 - 2\partial^2 \epsilon_{xy}/\partial x \partial y \qquad (2.9\text{-}3)$$

$$\partial^2 \epsilon_{yy}/\partial z^2 + \partial^2 \epsilon_{zz}/\partial y^2 - 2\partial^2 \epsilon_{yz}/\partial y \partial z \qquad (2.9\text{-}4)$$

$$\partial^2 \epsilon_{zz}/\partial x^2 + \partial^2 \epsilon_{xx}/\partial z^2 - 2\partial^2 \epsilon_{zx}/\partial z \partial x \qquad (2.9\text{-}5)$$

$$\partial^2 \epsilon_{zx}/\partial x \partial y + \partial^2 \epsilon_{xy}/\partial x \partial z - \partial^2 \epsilon_{yz}/\partial x^2 - 2\partial^2 \epsilon_{xx}/\partial y \partial z \qquad (2.9\text{-}6)$$

$$\partial^2 \epsilon_{xy}/\partial y \partial z + \partial^2 \epsilon_{yz}/\partial y \partial x - \partial^2 \epsilon_{zx}/\partial y^2 - 2\partial^2 \epsilon_{yy}/\partial z \partial x \qquad (2.9\text{-}7)$$

$$\partial^2 \epsilon_{yz}/\partial z \partial x + \partial^2 \epsilon_{zx}/\partial z \partial y - \partial^2 \epsilon_{xy}/\partial z^2 - 2\partial^2 \epsilon_{zz}/\partial x \partial y \qquad (2.9\text{-}8)$$

Cylindrical Co-ordinates, (r,θ,z)

$$\partial^2 \epsilon_{rr}/\partial z^2 + \partial^2 \epsilon_{zz}/\partial r^2 - 2\partial^2 \epsilon_{rz}/\partial z \partial r \qquad (2.9\text{-}9)$$

$$\partial^2 \epsilon_{\theta\theta}/\partial z^2 + \frac{1}{r^2} \partial^2 \epsilon_{zz}/\partial \theta^2 + \frac{1}{r} \partial \epsilon_{zz}/\partial r - \frac{2}{r} [\partial^2 \epsilon_{\theta z}/\partial z \partial \theta + \partial \epsilon/\partial z] \qquad (2.9\text{-}10)$$

$$\partial^2 \epsilon_{\theta\theta}/\partial r^2 + \frac{1}{r^2} \partial^2 \epsilon_{rr}/\partial \epsilon^2 + \frac{2}{r} \partial \epsilon_{\theta\theta}/\partial r - \frac{1}{r} \partial \epsilon_{rr}/\partial r$$

$$- 2[\frac{1}{r} \partial^2 \epsilon_{r\theta}/\partial r \partial \theta + \frac{1}{r^2} \partial \epsilon_{r\theta}/\partial \theta] \qquad (2.9\text{-}11)$$

$$\frac{1}{r} \partial^2 \epsilon_{zz}/\partial r \partial \theta - \frac{1}{r^2} \partial \epsilon_{zz}/\partial \theta - \partial(\frac{1}{r} \partial \epsilon_{zr}/\partial \theta + \partial \epsilon_{\theta z}/r$$

$$- \partial \epsilon_{r\theta}/\partial z)/\partial z - \partial(\epsilon_{\theta z}/r)/\partial z \qquad (2.9\text{-}12)$$

$$\frac{1}{r}\, \partial^2 \epsilon_{rr}/\partial\theta\partial z - \partial(\frac{1}{r}\, \partial\epsilon_{zr}/\partial\theta - \partial\epsilon_{\theta z}/\partial r$$

$$+ \partial\epsilon_{\theta r}/\partial z)/\partial r - \partial^2 \epsilon_{\theta z}/\partial r^2 + \frac{2}{r}\, \partial\epsilon_{r\theta}/\partial z \qquad (2.9\text{-}13)$$

$$\partial^2 \epsilon_{\theta\theta}/\partial r\partial z - \frac{1}{r}\, \partial\epsilon_{rr}/\partial z + \frac{1}{r}\, \partial\epsilon_{\theta\theta}/\partial z - \frac{1}{r}\, \partial(\epsilon_{\theta z}/r)/\partial\theta$$

$$+ \frac{1}{r}\, \partial[\, -\frac{1}{r}\, \partial\epsilon_{zr}/\partial\theta + \partial\epsilon_{\theta z}/\partial r + \partial\epsilon_{r\theta}/\partial z] \qquad (2.9\text{-}14)$$

2.10 PROBLEMS

1. Given the displacement components

$$u_x = Cx(y+z)^2$$
$$u_y = Cy(z+x)^2$$
$$u_z = Cz(x+y)^2$$

where C is a constant, find (a) the components of linear strains, (b) the components of the rotation tensor and (c) the principal elongation at a point (1,1,1).

Ans: (a)

$$\epsilon_{xx} = C(y+z)^2, \quad \epsilon_{xy} = C[x(y+z) + y(z+x)],$$
$$\epsilon_{yy} = C(z+x)^2, \quad \epsilon_{yz} = C[y(z+x) + x(y+z)],$$
$$\epsilon_{zz} = C(x+y)^2, \quad \epsilon_{zx} = C[z(x+y) + x(y+z)].$$

(b)

$$\omega_{xx} = \omega_{yy} = \omega_{zz} = 0$$

$$\omega_{xy} = C[x(y+z) - y(z+x)]$$

$$\omega_{yz} = C[y(z+x) - z(x+y)]$$

$$\omega_{zx} = C[z(x+y) - x(y+z)]$$

(c) at point (1,1,1):

$$|\bar{\epsilon}| = C \begin{vmatrix} 1-\epsilon & 4 & 4 \\ 4 & 1-\epsilon & 4 \\ 4 & 4 & 1-\epsilon \end{vmatrix} = 0$$

$$\epsilon_{11} = \epsilon_1 = 9 \qquad \epsilon_{22} = \epsilon_2 = -3 \qquad \epsilon_{33} = \epsilon_3 = -3$$

2. Show by direct differentiation of the strain-displacement relations Eqs.(2.2-6) that the compatibility conditions are the necessary conditions for finding a continuous single-valued displacement field.

3. Find the special state of strain that is derived from a body deformation symmetrically with respect to the origin of the co-ordinate system,

Ans: $u_R = u_R(R,0,0)$, $u_\theta = u_\phi = 0$

$$u_x = \frac{x}{R} u_R, \qquad u_y = \frac{y}{R} u_R, \qquad u_z = \frac{z}{R} u_R$$

$$\epsilon_{xx} = u_R/R + (x^2/R)\ d(u_R/R)/dR$$

$$\epsilon_{yy} = u_R/R + (y^2/R)\ d(u_R/R)/dR$$

$$\epsilon_{zz} = u_R/R + (z^2/R)\ d(u_R/R)/dR$$

$$\epsilon_{xy} = (2xy/R)\ d(u_R/R)/dR$$

$$\epsilon_{yz} = (2yz/R)\ d(u_R/R)/dR$$

$$\epsilon_{zx} = (2zx/R)\ d(u_R/R)/dR$$

4. Find all components of the strain, with respect to the rectangular co-ordinate, that are derived from a displacement that is axi-symmetry, i.e., symmetrical about the Oz axis:

$$u_r = u_r(r,z), \qquad u_\theta = 0, \qquad u_z = u_z$$

Ans:

$$u_x = x\ u_r/r, \qquad u_y = y\ u_r/r, \qquad u_z = u_z$$

$$\epsilon_{xx} = x\ u_r/r + x^2\ [\partial(x\ u_r/r)/\partial r]/r$$

$$\epsilon_{yy} = y\ u_r/r + y^2\ [\partial(y\ u_r/r)/\partial r]/r$$

$$\epsilon_{zz} = \partial u_z/\partial z$$

$$\epsilon_{xy} = 2xy\ [\partial(z\ u_r/r)/\partial r]/r$$

$$\epsilon_{yz} = (y/r)\ \partial u_z/\partial r + y\ \partial(y u_r/r)/\partial z$$

$$\epsilon_{zx} = (x/r)\ \partial u_z/\partial r + x\ \partial(x u_r/r)/\partial z$$

5. Give some reasons why the formulas in Eq.(2.10-1) will be valid for small strains only.

$$\epsilon_{xx} = \partial u_x/\partial x, \quad \epsilon_{yy} = \partial u_y/\partial y, \quad \epsilon_{zz} = \partial u_z/\partial z \qquad (2.10\text{-}1a)$$

$$\epsilon_{xy} = [\partial u_x/\partial y + \partial u_y/\partial x]/2 \qquad (2.10\text{-}1b)$$

$$\epsilon_{yz} = [\partial u_y/\partial z + \partial u_z/\partial y]/2 \qquad (2.10\text{-}1c)$$

$$\epsilon_{zx} = [\partial u_x/\partial z + \partial u_z/\partial x]/2 \qquad (2.10\text{-}1d)$$

6. The displacement field of a body is:

$$u_x = c_1 x, \quad u_y = c_2 y, \quad u_z = c_3 z$$

(a) Find the components ϵ_{ij} of the strain matrix, and the value of the three invariants of the state of strain if the constants c_1, c_2, and c_3 are so small that their squares and products are negligible,

(b) What is the value of the volumetric strain ϵ_v?

7. Solve Problem (2.10-6) for a displacement field given by

$$u_1 = c_1 x_2, \qquad u_2 = u_3 = 0.$$

Draw sketches showing a cubic element at a point, and with its edges parallel to the references axes, before and after transformation.

8. Let the expressions of the displacement field of a certain body be:

$$u_x = C(2x + y^2), \; u_y = C(x^2 - 3y^2), \; u_z = 0.$$

where $C = 10^{-2}$.

(a) Show the distorted shape of a two-dimensional element of area whose sides dx and dy are initially parallel to the coordinate axes; the element is located at a point M whose coordinates are $(2, 1/\sqrt{3}, 0)$.

(b) Determine the coordinates of M after transformation.

(c) Decompose the gradient of the displacement field at M into its symmetric and antisymmetric parts, i.e. find ϵ_{ij} and ω_{ij} with respect to the x,y,z axes.

(d) Find the angle of rotation and the cylindrical dilatation of the two elements dx and dy.

9. In Problem 2.10-8, compute the strain $\epsilon_{x'y'}$ of an element x'y' whose direction cosines are ($1/\sqrt{2}$, $1/\sqrt{2}$, ~~$1/\sqrt{3}$~~ 0). What are the principal directions and the principal strains?

10. Given the displacement field

$$u_1 = cx_1(x_2 + x_3) \; , \; u_2 = cx_2(x_3 + x_1) \; , \; u_3 = cx_3(x_1 + x_2)$$

where c is a small constant:

(a) Find the components of the linear strain, ϵ_{ij}.

(b) Find the components of the rotation, ω_{ij}.

11. The components of linear strain in a body are given by:

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & 0 & -cy \\ 0 & 0 & cx \\ -cy & cx & 0 \end{bmatrix}$$

where c is a constant. Find the principal strains and the principal directions at the point (1, 2, 0).

12. Determine the volumetric strain ϵ_v for the following state of strain:

$$[\epsilon_{ij}] = \begin{bmatrix} 0.5 & 1 & 0 \\ 1 & 2 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}$$

Compare the result to the unit change of volume $\Delta V/V_o$ and to the first invariant.

13. In a two-dimensional state of strain.

$$\epsilon_{xx} = 800 \times 10^{-6},\ \epsilon_{yy} = 100 \times 10^{-6},\ \epsilon_{xy} = -800 \times 10^{-6}.$$

Find the magnitude and direction of the principal strains, ϵ_1 and ϵ_2, both analytically and through the use of Mohr's diagram. Draw a sketch showing the deformation of a unit square with edges initially along OX and OY .

14. If

$$\epsilon_{xx} = -800 \times 10^{-6}, \ \epsilon_{yy} = -200 \times 10^{-6}, \ \epsilon_{xy} = -600 \times 10^{-6},$$

show in a suitable sketch the position of the axes with which the maximum shearing strain is associated.

15. Are the following states of strain possible?

(a) $\epsilon_{xx} = C(x^2+y^2)$
$\epsilon_{yy} = Cy^2$
$\epsilon_{xy} = 2Cxy$
$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$

(b) $\epsilon_{xx} = Cz(x^2+y^2)$
$\epsilon_{yy} = Cy^2z$
$\epsilon_{xy} = 2Cxyz$
$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$

where C is a constant.

16. Show by differentiation of the strain-displacement relations that the compatibility relations are necessary conditions for the existence of continuous single-valued displacements.

17. Establish by differentiation a set of compatibility relations involving both the ϵ_{ij}'s and the ω_{ij}'s.

18. Derive the equations of equilibrium in terms of displacements.

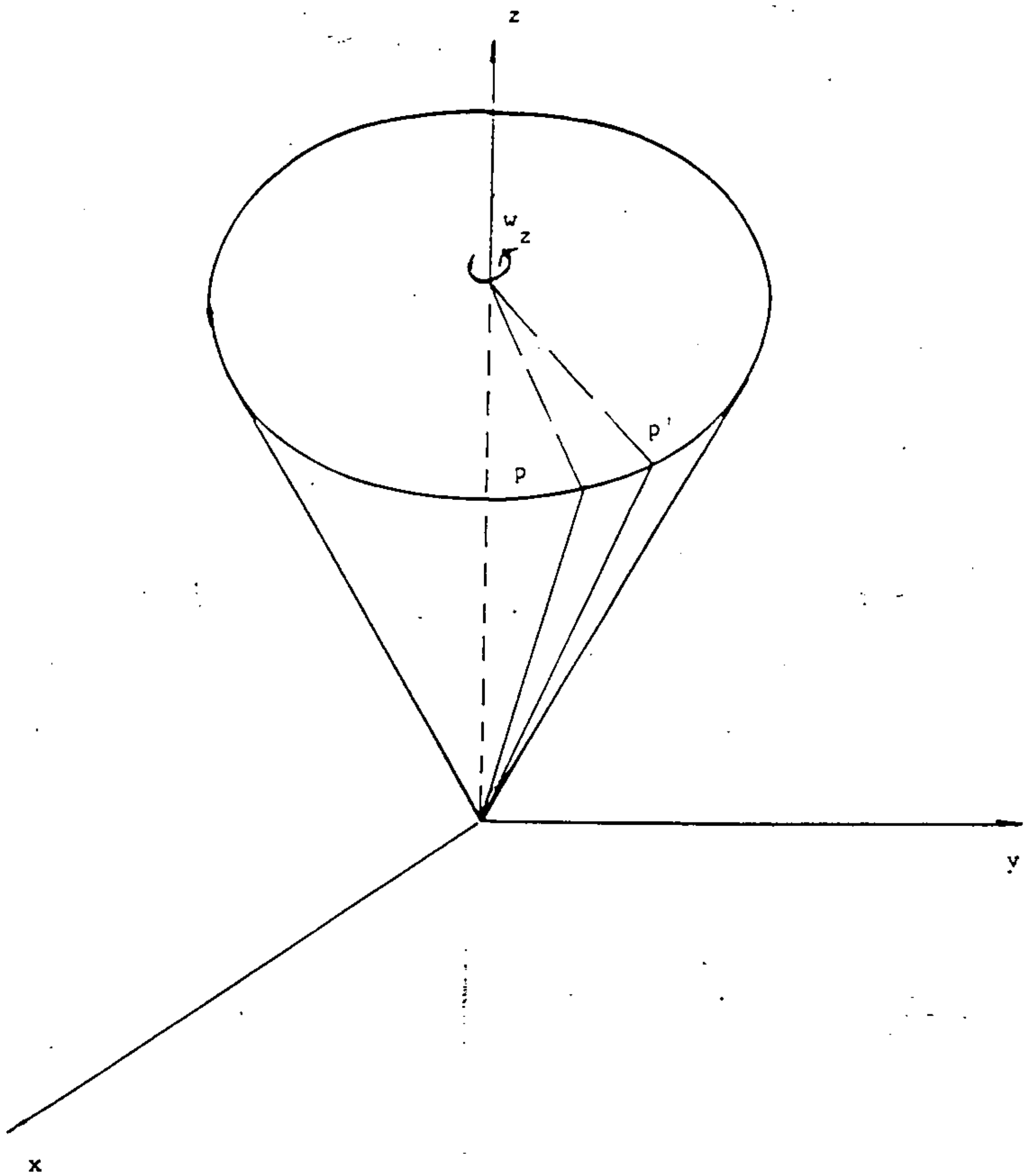

Fig. 2.1-2 Rigid body rotation about z-axis

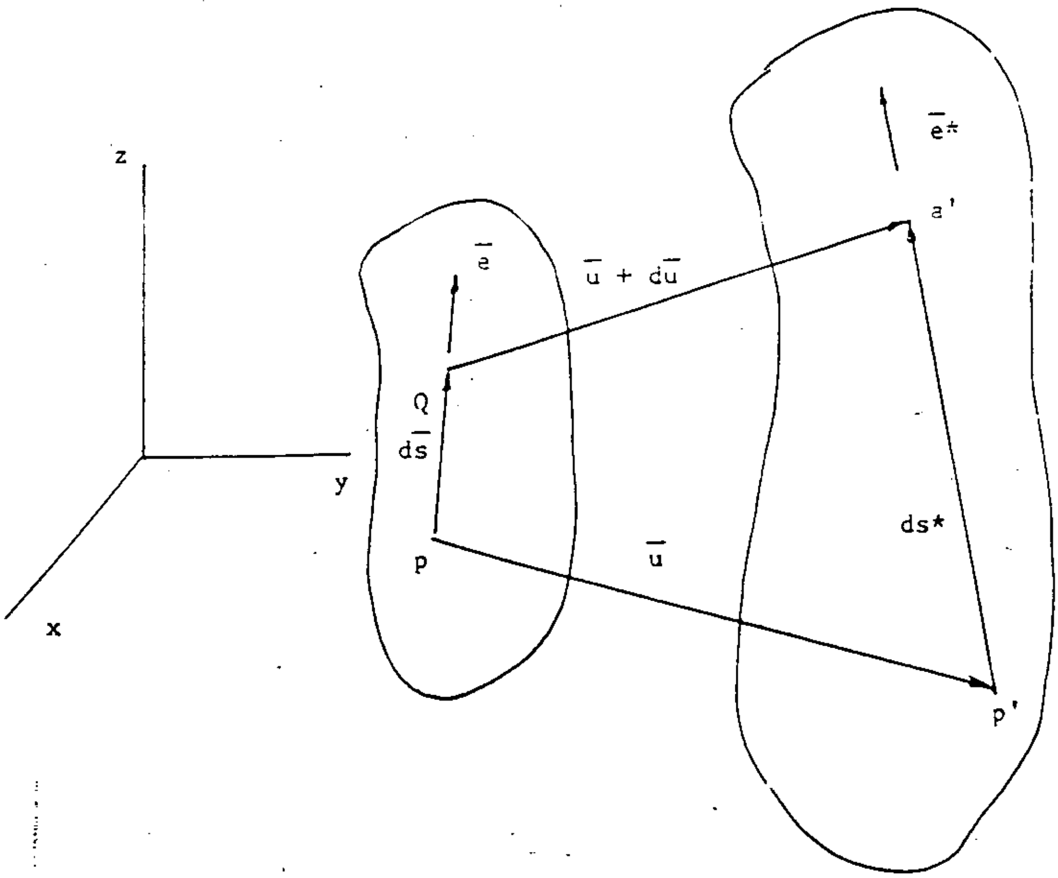

Fig.2.1-3 Deformation

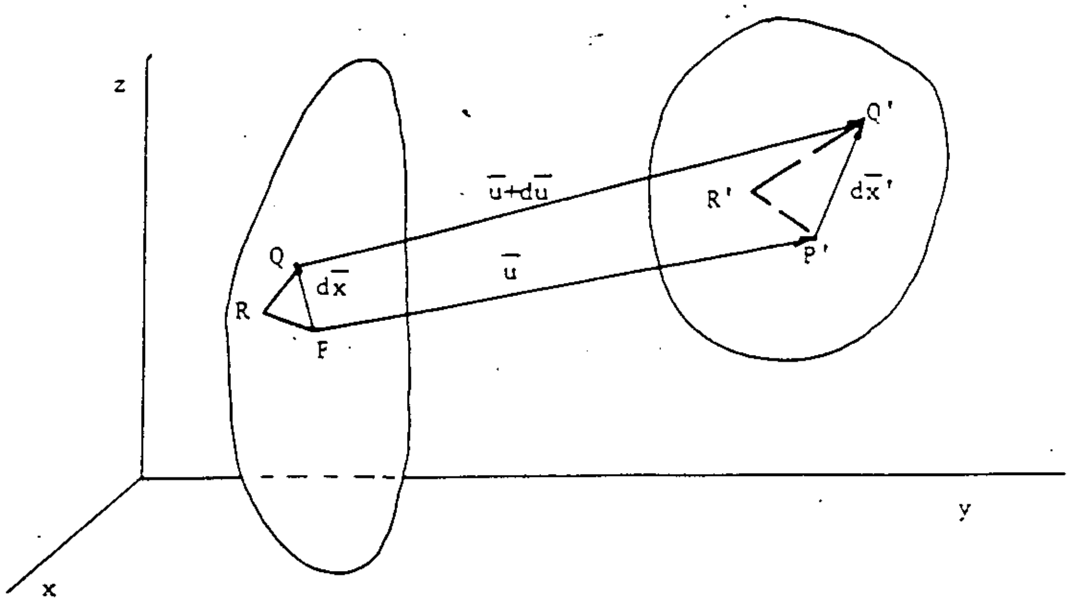

Fig.2.1-1 Displace and deformation

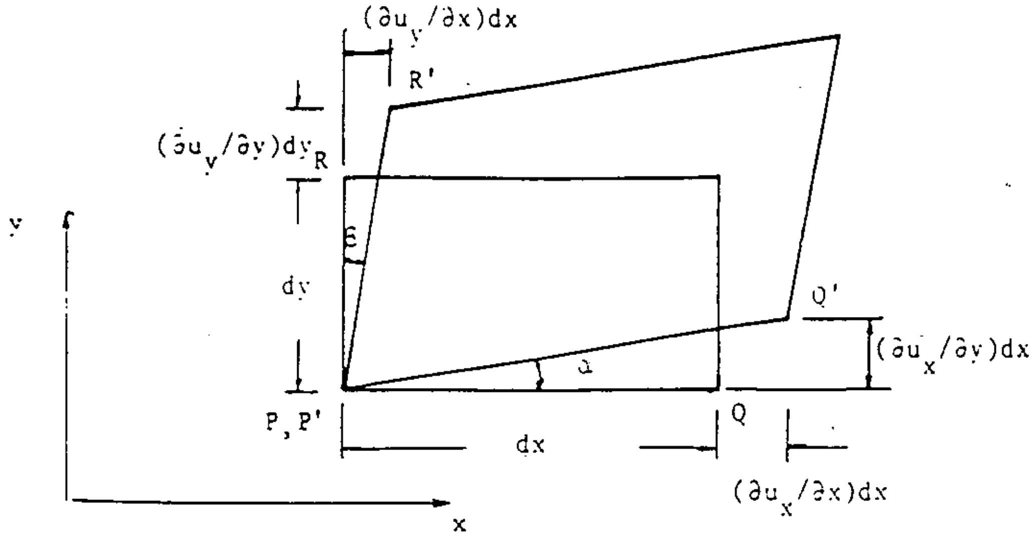

Fig.2.2-1 Change of shape and dimension of an element