تئوری الاستیسیته

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I. THEORY OF STRESS

1.1 STATE OF STRESS AT A POINT.

Traction: stress vector

Consider a material body under applied loading, in equilibrium or in motion. Internal reactions exist at internal surfaces if the body is separated into two parts, e.g. separated by a plane-cut n-n, The resultant action-reaction are equal in magnitude and opposite in direction by virtue of Newton's third law. For a continuously extended material in which the internal force over the internal area near the neighborhood of a material point is called the traction, or stress vector, at this point on the given internal plane, Fig.1.1-1.

$$\bar{T}^{(n)} = \lim_{\delta A \to 0} \frac{\Delta \bar{F}^{(n)}}{\Delta A} \qquad (1.1\text{-}1)$$

It is clear that the total force vector acting on ΔA, $\Delta \bar{F}^{(n)}$ can be broken down into one normal and two shear components, mutually perpendicular to each other. The stress vector obtained in Eq.(1.1-1) is a point function on the plane-cut n-n. Note that

$$\bar{T}^{(n)} = - \bar{T}^{(-n)} \qquad (1.1\text{-}2)$$

When the traction over any closed surface gives rise to vanishing resultant force and moment, the system of stress vector is said to be self-equilibrating.

1.2 CAUCHY FORMULA. CHARACTERIZATION OF THE STATE OF STRESS AT A POINT.

Let a plane-cut pass through the same point P on the plane-cut n-n, a new stress vector $\bar{T}^{(n')}$ is obtained at point P, Fig.1.2-1. The stress vector is seen to depend on the orientation of the plane-cut through a point. This requires an investigation as to how many plane cuts does it need to have in order to know the stress vector for all plane cuts.

Consider now a tetrahedron with mass Δm, acceleration $\bar{a}$, in the neighborhood of point P as represented in Fig.1.2-2. The internal planes on each of the plane cuts are denoted as $\Delta\bar{F}^{(-x)}$, $\Delta\bar{F}^{(-y)}$, $\Delta\bar{F}^{(-z)}$, $\Delta\bar{F}^{(n)}$ with the superscript indicating the normal vector of the plane-cut. Applying Newton's second law and dividing by the area ΔA_n, it can be easily obtained at P, in the limit $\Delta A_n \to 0$, the following relation between the traction $\bar{T}^{(n)}$ and $\bar{n}$:

$$\bar{T}^{(n)} = \bar{n} \cdot [\bar{e}_x \bar{T}^{(x)} + \bar{e}_y \bar{T}^{(y)} + \bar{e}_z \bar{T}^{(z)}] \qquad (1.2\text{-}1)$$

where $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$ are units vectors in the x,y,z axes, respectively. The relation given in Eq.(1.2-1) is referred to as the Cauchy formula. Writing the traction vectors in their component form,

$$\bar{T}^{(x)} = \sigma_{xx}\bar{e}_x + \sigma_{xy}\bar{e}_y + \sigma_{xz}\bar{e}_z \qquad (1.2\text{-}2)$$

$$\bar{T}^{(y)} = \sigma_{yz}\bar{e}_x + \sigma_{yy}\bar{e}_y + \sigma_{yz}\bar{e}_z \qquad (1.2\text{-}3)$$

$$\bar{T}^{(z)} = \sigma_{zx}\bar{e}_x + \sigma_{zy}\bar{e}_y + \sigma_{zz}\bar{e}_z \qquad (1.2\text{-}4)$$

then Eq.(1.2-1), i.e. the tractions on a plane with outer unit normal $\bar{n}$ becomes

$$T_x^{(n)} = \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \quad (1.2\text{-}5)$$

$$T_y^{(n)} = \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z \quad (1.2\text{-}6)$$

$$T_z^{(n)} = \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z \quad (1.2\text{-}7)$$

or

$$\bar{T}^{(n)} = \tilde{\sigma} \cdot \bar{n} \quad \rightarrow \quad T_i^{(n)} = \sigma_{ji} n_j \quad (1.2\text{-}8)$$

where

$$\bar{n} = n_x \bar{e}_x + n_y \bar{e}_y + n_z \bar{e}_z \quad (1.2\text{-}9)$$

Cauchy formula stipulates that the state of stress at a point is completely characterized by stress vectors, or tractions, on three mutually perpendicular planes as the traction on any plane of given outer unit normal can be obtained from Eqs. (1.2-5) to (1.2-7).

1.3 NOTATION.

Rectangular Co-ordinates (x,y,z): Fig.1.3-1

$$[\tilde{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (1.3\text{-}1)$$

Cylindrical Co-ordinates, (r, θ, z): Fig.1.3-2

$$[\bar{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta r} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix} \tag{1.3-2}$$

Spherical Co-ordinates (R, θ, ϕ), Fig.1.3-3

$$[\bar{\sigma}] = \begin{bmatrix} \sigma_{RR} & \sigma_{R\theta} & \sigma_{R\phi} \\ \sigma_{\theta R} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi R} & \sigma_{\phi R} & \sigma_{\phi\phi} \end{bmatrix} \tag{1.3-3}$$

$\sigma_{\phi r}$

1.4 PRINCIPAL STRESSES AND DIRECTIONS

The stress tensor is a symmetric second rank tensor and the direction cosines of the principal direction $\bar{n}$ is determined by using Eq.(A.1-45) or (A.1-46)

$$(\sigma_{xx}-\sigma)n_x + \sigma_{yx}n_y + \sigma_{zx}n_z = 0 \tag{1.4-1}$$

$$\sigma_{yx}n_x + (\sigma_{yy}-\sigma)n_y + \sigma_{yz}n_z = 0 \tag{1.4-2}$$

$$\sigma_{zx}n_x + \sigma_{zy}n_y + (\sigma_{zz}-\sigma)n_z = 0 \tag{1.4-3}$$

For non-trivial solution, the determinant of coefficients vanish, i.e.,

$$\begin{vmatrix} \sigma_{xx}-\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\sigma \end{vmatrix} = 0 \tag{1.4-4}$$

or

$$\sigma^3 - I_\sigma \sigma^2 + II_\sigma \sigma - III_\sigma = 0 \qquad (1.4\text{-}5)$$

from which the roots σ_I, σ_{II}, σ_{III} are obtained. They are the principal stresses and upon sustitution of these values in Eqs.(1.4-1) to (1.4-3), the corresponding principal directions can be obtained.

1.5 STRESS INVARIANTS. STRESS DEVIATOR

Spherical Part of Stress Tensor σ:

$$\sigma = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 \qquad (1.5\text{-}1)$$

Stress Deviator $\bar{s}$:

$$s_{xx} = \sigma_{xx} - \sigma, \qquad (1.5\text{-}2)$$

$$s_{yy} = \sigma_{yy} - \sigma, \qquad (1.5\text{-}3)$$

$$s_{zz} = \sigma_{zz} - \sigma, \qquad (1.5\text{-}4)$$

$$s_{xy} = \sigma_{xy}, \quad s_{yz} = \sigma_{yz}, \quad s_{zx} = \sigma_{zx} \qquad (1.5\text{-}5)$$

Stress Invariants I_σ, II_σ, III_σ:

$$I_\sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_I + \sigma_{II} + \sigma_{III} \qquad (1.5\text{-}6)$$

$$II_\sigma = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$$
$$= \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_{III}\sigma_I \qquad (1.5\text{-}7)$$

$$III_\sigma = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{yz}\sigma_{zx}$$
$$= \sigma_I\sigma_{II}\sigma_{III} \qquad (1.5\text{-}8)$$

where σ_I, σ_{II}, σ_{III} are the principal stresses.

*1.6 STRESS TRANSFORMATION, Fig.A.1-2

Let a_{ij} be the direction cosines between two orthogonal co-ordinate systems $\bar{x}$ and $\bar{x}'$, then

$$\sigma'_{ij}(\bar{x}') = a_{ip}a_{jq}\sigma_{pq}(\bar{x}),\ i,j,p,q = 1,2,3 \quad (1.6\text{-}1)$$

or

$$\sigma'_{11} = \sigma_{11}a_{11}^2+\sigma_{22}a_{12}^2+\sigma_{33}a_{13}^2+2\sigma_{12}a_{11}a_{12}+2\sigma_{13}a_{11}a_{13}+2\sigma_{23}a_{12}a_{13} \quad (1.6\text{-}2)$$

$$\sigma'_{22} = \sigma_{11}a_{21}^2+\sigma_{22}a_{22}^2+\sigma_{33}a_{23}^2+2\sigma_{12}a_{21}a_{22}+2\sigma_{13}a_{21}a_{23}+2\sigma_{23}a_{22}a_{23} \quad (1.6\text{-}3)$$

$$\sigma'_{33} = \sigma_{11}a_{31}^2+\sigma_{22}a_{32}^2+\sigma_{33}a_{33}^2+2\sigma_{12}a_{31}a_{32}+2\sigma_{13}a_{31}a_{33}+2\sigma_{23}a_{32}a_{33} \quad (1.6\text{-}4)$$

$$\sigma'_{12} = (\sigma_{11}a_{11}+\sigma_{21}a_{12}+\sigma_{31}a_{13})a_{21}+(\sigma_{12}a_{11}+\sigma_{22}a_{12}+\sigma_{32}a_{13})a_{22} +(\sigma_{13}a_{11}+\sigma_{23}a_{12}+\sigma_{33}a_{13})a_{23} \quad (1.6\text{-}5)$$

$$\sigma'_{23} = (\sigma_{11}a_{21}+\sigma_{21}a_{22}+\sigma_{31}a_{23})a_{31}+(\sigma_{12}a_{21}+\sigma_{22}a_{22}+\sigma_{32}a_{23})a_{32} +(\sigma_{13}a_{21}+\sigma_{23}a_{22}+\sigma_{33}a_{23})a_{33} \quad (1.6\text{-}6)$$

$$\sigma'_{13} = (\sigma_{11}a_{11}+\sigma_{21}a_{12}+\sigma_{31}a_{13})a_{31}+(\sigma_{12}a_{11}+\sigma_{22}a_{12}+\sigma_{32}a_{13})a_{32} +(\sigma_{13}a_{11}+\sigma_{23}a_{12}+\sigma_{33}a_{13})a_{33} \quad (1.6\text{-}7)$$

Cylindrical Co-ordinates: Fig. 1.6-1

a_{ij}	r	θ	z
x	cos θ	-sin θ	0
y	sin θ	cos θ	0
z	0	0	1

$$\sigma_{rr} = \cos^2\theta\ \sigma_{xx} + \sin 2\theta\ \sigma_{xy} + \sin^2\theta\ \sigma_{yy} \quad (1.6\text{-}8)$$

$$\sigma_{r\theta} = \sin 2\theta\ (\sigma_{yy}-\sigma_{xx})/2 + \cos 2\theta\ \sigma_{xy} \quad (1.6\text{-}9)$$

$$\sigma_{\theta\theta} = \sin^2\theta\ \sigma_{xx} - \sin 2\theta\ \sigma_{xy} + \cos^2\theta\ \sigma_{yy} \quad (1.6\text{-}10)$$

$$\sigma_{rz} = \cos\theta\ \sigma_{xz} + \sin\theta\ \sigma_{yz} \quad (1.6\text{-}11)$$

$$\sigma_{\theta z} = -\sin\theta\ \sigma_{xz} + \cos\theta\ \sigma_{yz} \quad (1.6\text{-}12)$$

$$\sigma_{zz} = \sigma_{zz} \quad (1.6\text{-}13)$$

and

$$\sigma_{xx} = \cos^2\theta\ \sigma_{rr} + \sin^2\theta\ \sigma_{\theta\theta} - \sin 2\theta\ \sigma_{r\theta} \quad (1.6\text{-}14)$$

$$\sigma_{yy} = \sin^2\theta\ \sigma_{rr} + \cos^2\theta\ \sigma_{\theta\theta} + \sin 2\theta\ \sigma_{r\theta} \quad (1.6\text{-}15)$$

$$\sigma_{zz} = \sigma_{zz} \quad (1.6\text{-}16)$$

$$\sigma_{xy} = [(\sin 2\theta)/2](\sigma_{rr} - \sigma_{\theta\theta}) + \cos 2\theta\ \sigma_{r\theta} \quad (1.6\text{-}17)$$

$$\sigma_{zx} = \cos\ \sigma_{zr} - \sin\theta\ \sigma_{z\theta} \quad (1.6\text{-}18)$$

$$\sigma_{zy} = \sin\theta\ \sigma_{zr} + \cos\theta\ \sigma_{z\theta} \quad (1.6\text{-}19)$$

Spherical Co-ordinates: Fig. 1.6-2

(1)

a_{ij}	R	θ	ϕ
x	$\sin\phi\cos\theta$	$-\sin\theta$	$\cos\phi\cos\theta$
y	$\sin\phi\sin\theta$	$\cos\theta$	$\cos\phi\sin\theta$
z	$\cos\phi$	0	$-\sin\phi$

$$\sigma_{RR} = (\sin^2\phi\cos^2\theta)\ \sigma_{xx} + (\sin^2\phi\sin^2\theta)\ \sigma_{yy} + \cos^2\phi\ \sigma_{zz}$$

$$+ \sin^2\phi\sin 2\theta\ \sigma_{xy} + \sin 2\phi\cos\theta\ \sigma_{xz} + \sin 2\phi\sin\theta\ \sigma_{yz} \quad (1.6\text{-}20)$$

$$\sigma_{\theta\theta} = \sin^2\theta\ \sigma_{xx} + \cos^2\theta\ \sigma_{yy} - \sin 2\theta\ \sigma_{xy} \quad (1.6\text{-}21)$$

$$\sigma_{\phi\phi} = (\cos^2\phi\cos^2\theta)\ \sigma_{xx} + (\cos^2\phi\sin^2\theta)\ \sigma_{yy} + \sin^2\phi\ \sigma_{zz}$$

$$+ \cos^2\phi \sin 2\theta\, \sigma_{xy} - \sin 2\phi \cos\theta\, \sigma_{xz} - \sin 2\phi \sin\theta\, \sigma_{yz} \quad (1.6\text{-}22)$$

$$\sigma_{R\theta} = -\tfrac{1}{2}(\sin\phi \sin 2\theta)\, \sigma_{xx} + \tfrac{1}{2}(\sin\phi \sin 2\theta)\, \sigma_{yy}$$

$$+ \sin\phi \cos 2\theta\, \sigma_{xy} - \cos\phi \sin\theta\, \sigma_{zx} + \cos\phi \cos\theta\, \sigma_{zy} \quad (1.6\text{-}23)$$

$$\sigma_{\theta\phi} = -\tfrac{1}{2}(\cos\phi \sin 2\theta)\, \sigma_{xx} + \tfrac{1}{2}(\cos\phi \sin 2\theta)\, \sigma_{yy}$$

$$+ \cos\phi \cos 2\theta\, \sigma_{xy} + \sin\phi \sin\theta\, \sigma_{xz} - \sin\phi \cos\theta\, \sigma_{yz} \quad (1.6\text{-}24)$$

$$\sigma_{R\phi} = \tfrac{1}{2}(\sin 2\phi \cos^2\theta)\, \sigma_{xx} + \tfrac{1}{2}(\sin 2\phi \sin^2\theta)\, \sigma_{yy} - \tfrac{1}{2}(\sin 2\phi)\sigma_{zz}$$

$$+ \tfrac{1}{2}\sin 2\phi \sin 2\theta\, \sigma_{xy} + \cos 2\phi \sin\theta\, \sigma_{yz}$$

$$+ \cos 2\phi \cos\theta\, \sigma_{zx} \quad (1.6\text{-}25)$$

(2)

a_{ij}	R	θ	ϕ
r	$\sin\phi$	0	$\cos\phi$
θ	0	1	0
z	$\cos\phi$	0	$-\sin\phi$

$$\sigma_{RR} = \sin^2\phi\, \sigma_{rr} + \cos^2\phi\, \sigma_{zz} + \sin 2\phi\, \sigma_{rz} \quad (1.6\text{-}26)$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta} \quad (1.6\text{-}27)$$

$$\sigma_{\phi\phi} = \cos^2\phi\, \sigma_{rr} + \sin^2\phi\, \sigma_{zz} - \sin 2\phi\, \sigma_{rz} \quad (1.6\text{-}28)$$

$$\sigma_{R\theta} = \sin\phi\, \sigma_{r\theta} + \cos\phi\, \sigma_{\theta z} \quad (1.6\text{-}29)$$

$$\sigma_{\theta\phi} = \cos\phi\, \sigma_{r\theta} - \sin\phi\, \sigma_{\theta z} \quad (1.6\text{-}30)$$

$$\sigma_{Rz} = \tfrac{1}{2}(\sin 2\phi)\, \sigma_{rr} - \tfrac{1}{2}(\sin 2\phi)\, \sigma_{zz} + \cos 2\phi\, \sigma_{rz} \quad (1.6\text{-}31)$$

*1 7 GEOMETRIC REPRESENTATION OF THE STATE OF STRESS AT A POINT.

Fig.1.7-1

Lame' Stress Ellipsoid

Let the Cartesian co-ordinate I,II,III be coincide with the principal axes of the stress tensor, then from Eqs.(1.6.1) to (1.6.4) we obtain:

$$[T_I^{(n)}/\sigma_I]^2 + [T_{II}^{(n)}/\sigma_{II}]^2 + [T_{III}^{(n)}/\sigma_{III}]^2 = 1 \qquad (1.7\text{-}1)$$

Eq. (1.7-1) represents an ellipsoid that is the locus of the end points of the traction vectors $\bar{T}^{(n)}$.

Cauchy's Stress Quadric, Fig.1.7-2

Let

N = normal component of traction vector

$$= \bar{T}^{(n)} \cdot \bar{n} \qquad (1.7\text{-}2)$$

and

$\bar{x}$ = any vector along $\bar{n}$, say OP

$$= |\overline{OP}|\ \bar{n} \qquad (1.7\text{-}3)$$

then

$$N\ |\bar{x}|^2 = [\bar{T}^{(n)} \cdot \bar{n}](\bar{x} \cdot \bar{x})$$

$$= \sigma_{ij}\ x_i x_j \qquad (1.7\text{-}4)$$

Consider the Cauchy's stress quadric

$$-f(x,y,z) = -\sigma_{ij}x_ix_j \pm k_0^2 = 0, \quad k_0^2 = \text{constant} \qquad (1.7\text{-}5)$$

Let $\sigma_{ij} = \sigma_{ji}$ and x_i be spanned in the principal directions of the stress tensor, the Cauchy stress quadric then takes the form:

$$\sigma_1(x_1)^2 + \sigma_2(x_2)^2 + \sigma_3(x_3)^2 = \pm k_0^2 \qquad (1.7\text{-}6)$$

Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then Eq.(1.7-6) represents

(a) an ellipsoid, if

$\sigma_1 > \sigma_2 > \sigma_3 > 0, \quad (+k_0^2);$

$0 > \sigma_1 > \sigma_2 > \sigma_3, \quad (-k_0^2)$

(b) an ellipsoid of revolution if principal stresses are of the same sign and two of them having same values, i.e.,

$\sigma_1 = \sigma_2$, or $\sigma_2 = \sigma_3$, or $\sigma_3 = \sigma_1$.

(c) a sphere if $\sigma_1 = \sigma_2 = \sigma_3$

(d) a hyperboloid of one sheet if

$\sigma_1 \geq \sigma_2 > 0, \quad \sigma_3 < 0, \quad (+k_0^2)$

(e) a hyperboloid of two sheets if

$\sigma_1 \geq \sigma_2 > 0, \quad \sigma_3 < 0, \quad (-k_0^2)$

<u>Mohr's Circle</u>, Fig.1.7-3

Let $\vec{x}$ be spanned in the principal directions 1,2, and 3:

$$N = \vec{T}^{(n)} \cdot \vec{n}$$
$$= \sigma_1 n_x^2 + \sigma_2 n_y^2 + \sigma_3 n_z^2 \qquad (1.7\text{-}7)$$

$$\bar{T} \cdot \bar{T} = N^2 + S^2 \qquad (1.7\text{-}8)$$

$$n_x^2 + n_y^2 + n_z^2 = 1 \qquad (1.7\text{-}9)$$

from which

$$n_x^2 = [(\sigma_2 - N)(\sigma_3 - N) + S^2]/[(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)] \qquad (1.7\text{-}10)$$

$$n_y^2 = [(\sigma_3 - N)(\sigma_1 - N) + S^2]/[(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)] \qquad (1.7\text{-}11)$$

$$n_z^2 = [(\sigma_1 - N)(\sigma_2 - N) + S^2]/[(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)] \qquad (1.7\text{-}12)$$

For given N and S:

$$[N - (\sigma_2 + \sigma_3)/2]^2 + S^2 = r_x^2, \qquad n_x = \text{constant} \qquad (1.7\text{-}13)$$

$$[N - (\sigma_3 + \sigma_1)/2]^2 + S^2 = r_y^2, \qquad n_y = \text{constant} \qquad (1.7\text{-}14)$$

$$[N - (\sigma_1 + \sigma_2)/2]^2 + S^2 = r_z^2, \qquad n_z = \text{constant} \qquad (1.7\text{-}15)$$

where

$$r_x^2 = n_x^2\,(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + [(\sigma_2 - \sigma_3)/2]^2 \ge 0 \qquad (1.7\text{-}16)$$

$$r_y^2 = n_y^2\,(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1) + [(\sigma_3 - \sigma_1)/2]^2 \ge 0 \qquad (1.7\text{-}17)$$

$$r_z^2 = n_z^2\,(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2) + [(\sigma_1 - \sigma_2)/2]^2 \ge 0 \qquad (1.7\text{-}18)$$

(a) Any given state of stress at a point is bounded between the three circles shown in Fig. 1.7-4, i.e. the shaded area. The largest and smallest normal stresses are σ_1 and σ_3, respectively.

(b) The absolute maximum shear stress is $\tau_{max} = (\sigma_1 - \sigma_3)/2$

(c) The location of the center of the circles depends only upon the spherical part of the stress tensor and the deviatoric part of the stress tensor governs the size of the circles.

1.8 EQUILIBRIUM EQUATIONS, Fig.1.8-1

Force equilibrium condition:

$$\iiint_V (\vec{f} - \rho\ddot{\vec{u}})dV + \iint_S \vec{T}^{(n)}dS = 0$$
$$\rightarrow \iiint_V (f_i - \rho\ddot{u}_i)\ dV + \iint_S T_i\ dS = 0 \qquad (1.8\text{-}1)$$

yields equations of equilibrium:

$$\vec{\nabla}\ \vec{\sigma} + \vec{f} = \rho\ddot{\vec{u}} \qquad \rightarrow \quad \sigma_{ji,j} + f_i = \rho\ddot{u}_i \qquad (1.8\text{-}2)$$

Moment equilibrium condition:

$$\iiint_V \vec{r} \times (\vec{f} - \rho\ddot{\vec{u}})dV + \iint_S \vec{r} \times \vec{T}^{(n)}dS = 0, \qquad \vec{r} = \text{position vector}$$
$$\rightarrow \iiint_V e_{ijk}x_j(f_k - \rho\ddot{u}_k) + \iint_S e_{ijk}x_jT_k = 0, \qquad (1.8\text{-}3)$$

leads to the symmetry in stress tensor:

$$\tilde{e} \cdot \tilde{\sigma} = 0 \quad \rightarrow \quad e_{ijk}\sigma_{jk} = 0 \quad \rightarrow \quad \sigma_{jk} = \sigma_{kj} \qquad (1.8\text{-}4)$$

where $\tilde{e}$ is permutation tensor defined in Eq.(4.1-5)

Rectangular Co-ordinates, (x,y,z), Fig.1.8-2

$$\partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + \partial\sigma_{zx}/\partial z + f_x = 0 \quad (= \rho\ \ddot{u}_x) \qquad (1.8\text{-}5)$$

$$\partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{zy}/\partial z + f_y = 0 \quad (= \rho\ \ddot{u}_y) \qquad (1.8\text{-}6)$$

$$\partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + f_z = 0 \quad (= \rho\ \ddot{u}_z) \qquad (1.8\text{-}7)$$

Cylindrical Co-ordinates, (r,θ,z), Fig.1.8-3

$$\partial(\sigma_{rr})/\partial r + \frac{1}{r}\,\partial\sigma_{r\theta}/\partial\theta + \partial\sigma_{rz}/\partial z + (\sigma_{rr} - \sigma_{\theta\theta})/r + f_r = 0\ (-\rho\ddot{u}_r) \qquad (1.8\text{-}8)$$

$$\partial(\sigma_{\theta r})/\partial r + \frac{1}{r}\,\partial\sigma_{\theta\theta}/\partial\theta + \partial\sigma_{\theta z}/\partial z + 2\sigma_{r\theta}/r + f_\theta = 0\ (-\rho\ddot{u}_\theta) \qquad (1.8\text{-}9)$$

$$\partial(\sigma_{zr})/\partial\theta + \frac{1}{r}\,\partial\sigma_{z\theta}/\partial\theta + r\,\partial\sigma_{zz}/\partial z + \sigma_{rz}/r + f_z = 0\ (-\rho\ddot{u}_z) \qquad (1.8\text{-}10)$$

Spherical Co-ordinates, (R,θ,ϕ), Fig.1.8-4

$$\partial\sigma_{RR}/\partial R + (\partial\sigma_{R\theta}/\partial\theta)/(R\sin\phi) + (\partial\sigma_{R\phi}/\partial\phi)/R$$
$$+ [\cot\phi\ \sigma_{r\phi} + (2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi})]/R + f_R = 0\ (\rho\ddot{u}_R) \qquad (1.8\text{-}11)$$

$$\partial\sigma_{R\theta}/\partial R + (\partial\sigma_{\phi\theta}/\partial\phi)/R + (\partial\sigma_{\theta\theta}/\partial\theta)/(R\sin\phi)$$
$$+ [3\sigma_{R\theta} + 2\cot\phi\ \sigma_{\phi\theta}]/R + f_\theta = 0\ (-\rho\ddot{u}_\theta) \qquad (1.8\text{-}12)$$

$$\partial\sigma_{R\phi}/\partial R + (\partial\sigma_{\phi\phi}/\partial\phi)/R + (\partial\sigma_{\phi\phi}/\partial\theta)/(R\sin\phi)$$
$$+ [3\sigma_{R\phi} + (\cot\phi)(\sigma_{\phi\phi} - \sigma_{\theta\theta})]/R + f_\phi = 0\ (-\rho\ddot{u}_\phi) \qquad (1.8\text{-}13)$$

1.9 PROBLEMS

1. Determine the principal stresses and directions for the state of stress at a point given as follows:

$$\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$$

(a) Using Eq. (1.1-45) we find

$$\begin{vmatrix} 2-\sigma & 4 & -6 \\ 4 & 2-\sigma & -6 \\ -6 & -6 & -15-\sigma \end{vmatrix} = 0$$

or

$$(\sigma - 9)(\sigma + 2)(\sigma + 18) = 0$$

The principal stresses are: $\sigma_1 = 9$, $\sigma_2 = -2$, $\sigma_3 = -18$

(b) Substituting each of the principal values in Eq.(1.1-1) and noting that $\bar{n}$ is a vector, we obtain:

$$\sigma_1 = 9, \quad \bar{n}_1 = (2/3, \; 2/3, \; -1/3)$$

$$\sigma_2 = -2, \quad \bar{n}_2 = (-1/\sqrt{2}, \; 1/\sqrt{2}, \; 0)$$

$$\sigma_3 = -18, \quad \bar{n}_3 = (1/\sqrt{18}, \; 1/\sqrt{18}, \; 4/\sqrt{18})$$

Note that $\bar{n}_1 \cdot \bar{n}_2 = \bar{n}_2 \cdot \bar{n}_3 = \bar{n}_3 \cdot \bar{n}_1 = 0$, i.e. the principal directions are mutually perpendicular.

2. Let σ_1, σ_2 and σ_3 be the principal stresses that span the stress space. Show that the equation

$$(s_1^2 + s_2^2 + s_3^2)/2 = k^2 \qquad 1,2,3 \text{ are principal directions}$$

$$s_i = \sigma_i - (\sigma_1 + \sigma_2 + \sigma_3)/3 \qquad i = 1,\ 2,\ 3$$

$$k = \text{constant}$$

represents the surface of a cylinder and the radius of this cylinder is $\rho = \sqrt{2}\,k$, Fig.1.9-1.

3. (a) Find the rectangular components of the traction vector $\hat{T}$ on the plane with unit normal vector $(1/\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3})$ and (b) determine the normal and tangential components of the traction vector if the state of stress at the point is given as:

$$\sigma = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}$$

Ans. (a) From Eq.(1.2-5) to (1.2-7)

$$T_x = 3\cdot(1/\sqrt{3}) + 1\cdot(1/\sqrt{3}) + 4\cdot(-1/\sqrt{3}) = 0$$

$$T_y = (1/\sqrt{3}) + 2\cdot(1/\sqrt{3}) - (-5)(1/\sqrt{3}) = 8/\sqrt{3}$$

$$T_z = 4\cdot(1/\sqrt{3}) + (-5)(1/\sqrt{3} + 0 = -1/\sqrt{3}$$

Hence:

$$\bar{T} = (0,\ 8/\sqrt{3},\ -1/\sqrt{3}), \qquad T = 65/3$$

(b) Normal and shear components are

$$T_N = \bar{n} \cdot \bar{T} = 0.3$$
$$T_S = (T^2 - T_S^2)^{1/2} = (38/3)^{1/2}$$

4. For the given stress field,

$$\sigma_{xx} = 3x^2 + 4xy - 8y^2$$
$$\sigma_{yy} = 2x^2 + xy + 3y^2$$
$$\sigma_{xy} = -x^2/2 - 6xy - 2y^2$$
$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$

(a) Does equilibrium exist in the absence of body force?

(b) What conditions does this stress field impose upon a region bounded between $0 \le y \le 1$ and $0 \le x \le 1$?

Ans.:(a) From Eqs.(1.8-5) to (1.8-7)

$$\partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y = 6x + 4y + (-6x-4y) = 0$$
$$\partial\sigma_{yx}/\partial x + \partial\sigma_{yy}/\partial y = -x - 6y + (x + 6y) = 0$$

Hence, static equilibrium conditions are satisfied in the absence of body forces.

(b) $x = 0$: $\sigma_{xx} = -8y^2$, $\qquad \sigma_{xy} = -2y^2$

$$x = 1: \quad \sigma_{xx} = 3 + 4y - 8y^2, \quad \sigma_{xy} = -1/2 - 6y - 2y^2$$
$$y = 0: \quad \sigma_{yy} = 2x^2, \quad \sigma_{yx} = -x^2/2$$
$$y = 1: \quad \sigma_{yy} = 2x^2 + x + 3, \quad \sigma_{yx} = -x^2/2 - 6x - 2$$

Note that only components of the traction vector appears on boundaries.

5. The stress field in a continuous body is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 0 & 2x \\ 0 & 1 & -2y \\ 2x & -2y & x \end{bmatrix} \qquad (1.9\text{-}1)$$

(a) Find the stress vector at a point $M:(1, \frac{1}{\sqrt{2}}, 0)$ on a plane with unit normal $\bar{n}:(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$

(b) What are the normal and tangential stresses acting on this plane?

(c) What are the principal stresses and the principal direction that corresponds to the smallest principal value at this point.

6. List boundary conditions in terms of tractions or as appropriate for the following loaded structure, Fig.1.9-2.

7. For the following stress tensors

(a) find the eigenvalues

(b) find the eigenvectors

(c) show that the eigenvectors are orthogonal

(d) find the transformation matrix required to go from the original coordinate system to the principal coordinate system.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

8. The stress field in a continuous body is given by:

$$[\sigma_{ij}] = 10^3 \begin{bmatrix} 1 & 0 & 2y \\ 0 & 1 & 4x \\ 2y & 4x & 1 \end{bmatrix} \text{ psi.}$$

Find the stress vector $\bar{T}$ at a point M (1, 1, 1), acting on a plane $x - y - z = -1$.

9. The state of stresses at a point is given by:

$$[\sigma_{ij}] = 10^2 \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \text{ psi.}$$

Find the magnitude and direction of the stress vector acting on a plane whose normal has direction cosines $(1/2, 1/2, 1/\sqrt{2})$; what are the normal and tangential stresses acting on this plane?

10. In a solid circular shaft subjected to pure torsion, the stress field is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & -Cy \\ 0 & 0 & Cx \\ -Cy & Cx & 0 \end{bmatrix}$$

where C is a constant. At the point whose coordinates are (1, 2, 4), find:

(a) the principal stresses

(b) the principal directions

(c) the maximum shearing stress and the plane on which it acts.

11. At a point M of a continuous body, the components of the stress tensor are:

$$[\sigma_{ij}] = 10^3 \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix} \text{ psi.}$$

(a) Find the principal stresses and the principal directions.

(b) Obtain the normal and tangential stresses on a plane whose normal has direction cosines $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$ with respect to the reference axes.

(c) Find the octahedral normal and shearing stresses.

(d) What are the invariants of the spherical and the deviatoric components of this stress tensor?

(e) What is the equation of the stress quadric?

12. Find the components of the stress tensor of Problem 1.9-11 in a system of coordinates whose axes have direction cosines

$$(0,\ 0,\ 1),\ (1/\sqrt{2},\ 1/\sqrt{2},\ 0),\ (1/\sqrt{2},\ -1/\sqrt{2},\ 0).$$

13. A very thin plate is uniformly loaded as shown, Fig.1.9-3. Among all the planes that are normal to the plane of the plate, which ones are the principal planes and what is the value of the stresses to which they are subjected?

14. For the following states of stress at a point, use Mohr's circle to obtain the magnitude and directions of the principal stresses:

(a) $\sigma_{11} = 4{,}000$ psi, $\sigma_{22} = 0$, $\sigma_{12} = 8{,}000$ psi
$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

(b) $\sigma_{11} = 14{,}000$ psi, $\sigma_{22} = 5{,}000$ psi, $\sigma_{12} = 6{,}000$ psi
$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

(c) $\sigma_{11} = 12{,}000$ psi, $\sigma_{22} = 5{,}000$ psi, $\sigma_{12} = 10{,}000$ psi
$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

15. If the stress tensor at a point within the body is given by

$$\begin{bmatrix} 1000 & 200 & 0 \\ 200 & -600 & -400 \\ 0 & -400 & 0 \end{bmatrix}$$

find the normal stress in the direction n, where $\bar{n}$ is the unit vector:

$$\bar{n} = \frac{2}{3}\,\bar{e}_x + \frac{2}{3}\,\bar{e}_y - \frac{1}{3}\,\bar{e}_z$$

16. Consider a body under a state of biaxial shear:

$$\begin{bmatrix} 0 & x_3 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_3 = z$$

Find the principal axes and the principal stresses at a point in the body with coordinate (0, 1, 2).

17. For the state of stress shown in Problem 16, find the stress tensor with reference to a coordinate system rotated 60° about the x_3 axis by the right-hand rule.

18. Find the spherical and deviatoric stress tensors from the stress tensor shown in Problem 9.

19. A stress field is given by:

$$\begin{aligned} \sigma_{xx} &= 2x^3 + y^2 & \sigma_{xy} &= z \\ \sigma_{yy} &= 3x^3 + 20 & \sigma_{xz} &= y \\ \sigma_{zz} &= 3y^2 + 3z^3 & \sigma_{yz} &= x^3 \end{aligned} \qquad (1.9\text{-}2)$$

What are the components of the body force required to insure equilibrium?

20. Show that the conditions on stresses at a point that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{zx} = \sigma_{xz}, \quad \sigma_{zy} = \sigma_{yz} \qquad (1.9\text{-}3)$$

continue to hold even if the element shown is in motion and has an angular acceleration like a rigid body.

21. Suppose an elastic material contains a large number of evenly distributed small magnetized particles, so that a magnetic field exerts on any element dx dy dz a moment μ dx dy dz about an axis parallel to the x axis. What modification will be needed in Eqs.(1.9-3)?

22. Consider an elastic solid acted upon by body forces that exert moments M per unit volume (as in the case of a polarized dielectric solid under the action of an electric field). Show that in this case, eqs. of equilibrium must be replaced by

$$\epsilon_{ijk} \sigma_{jk} + M_i = 0 \qquad (1.9\text{-}4)$$

What can be said in this case about the symmetry of the stress components?

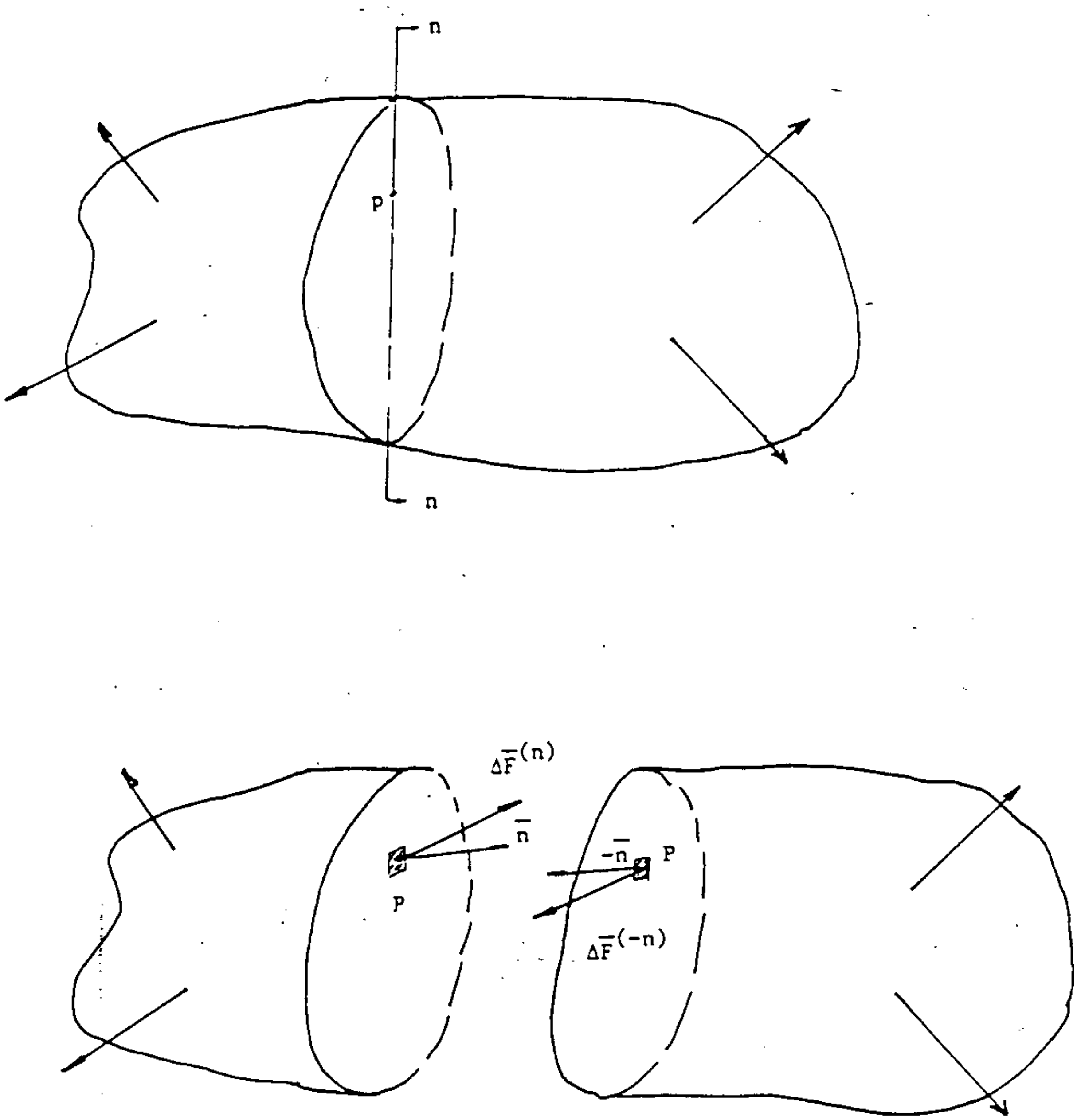

Fig. 1.1-1 Internal force acting at a point P

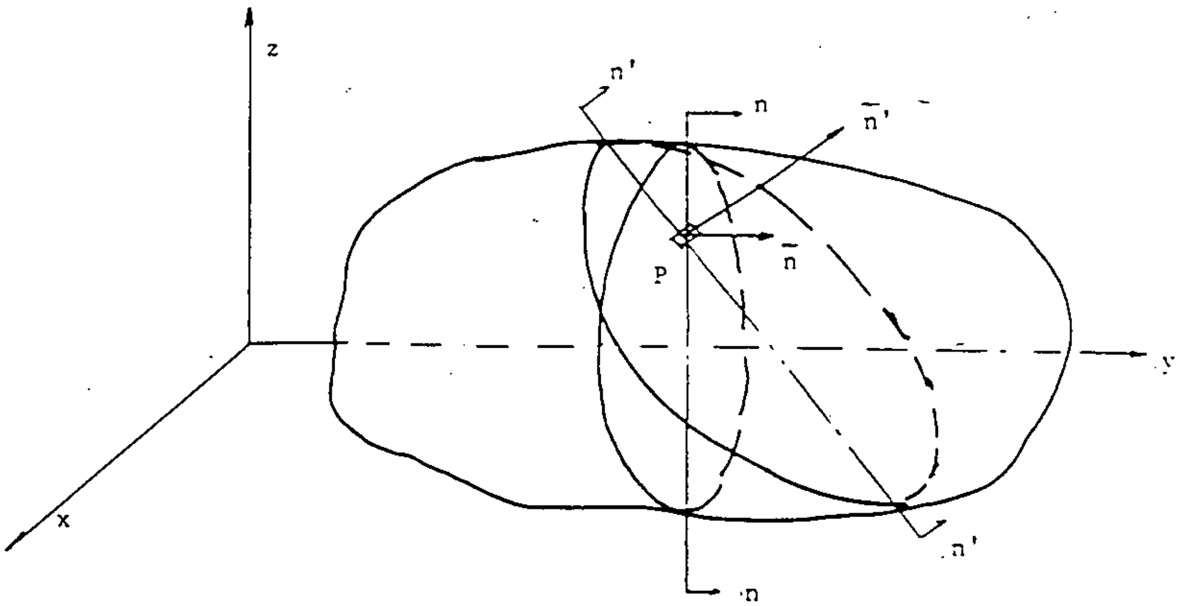

Fig.1.2-1 Differently oriented cut at point P

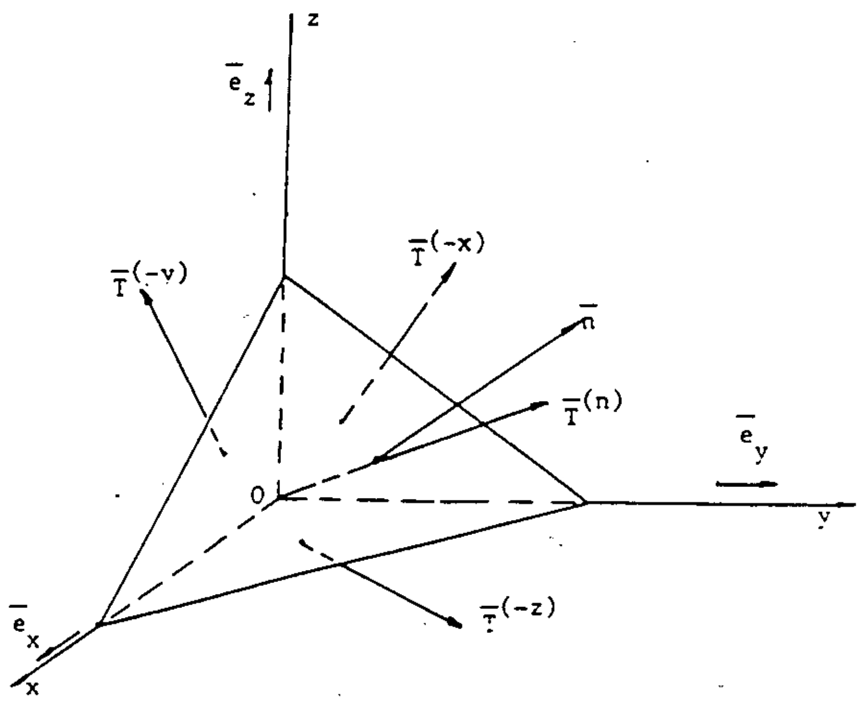

Fig.1.2-2 Stress vector $\bar{T}^{(n)}$ on a surface element with unit outer normal vector $\bar{n}$

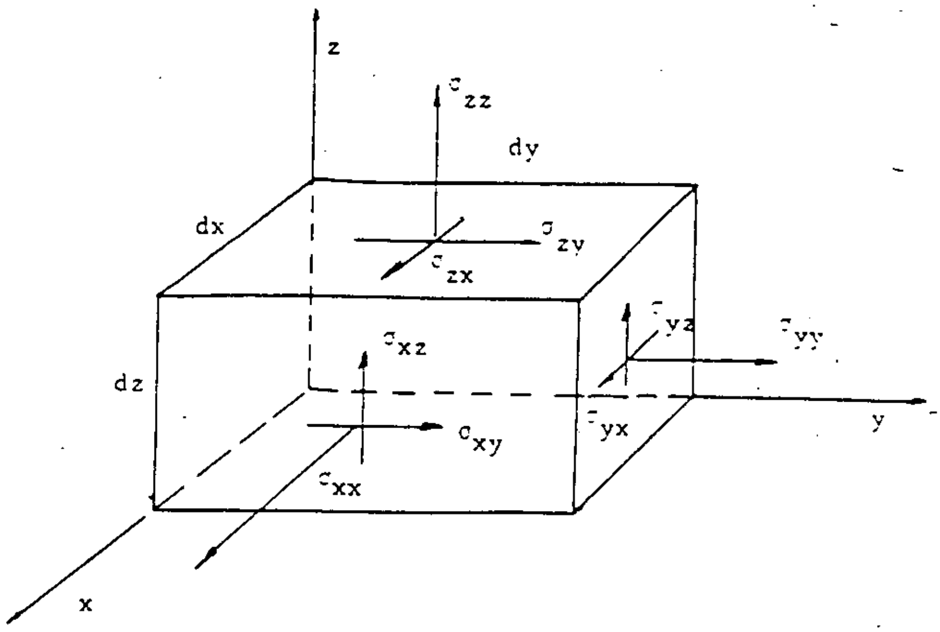

Fig.1.3-1 Stress tensor on a material element at a point (stresses on negative faces not shown)

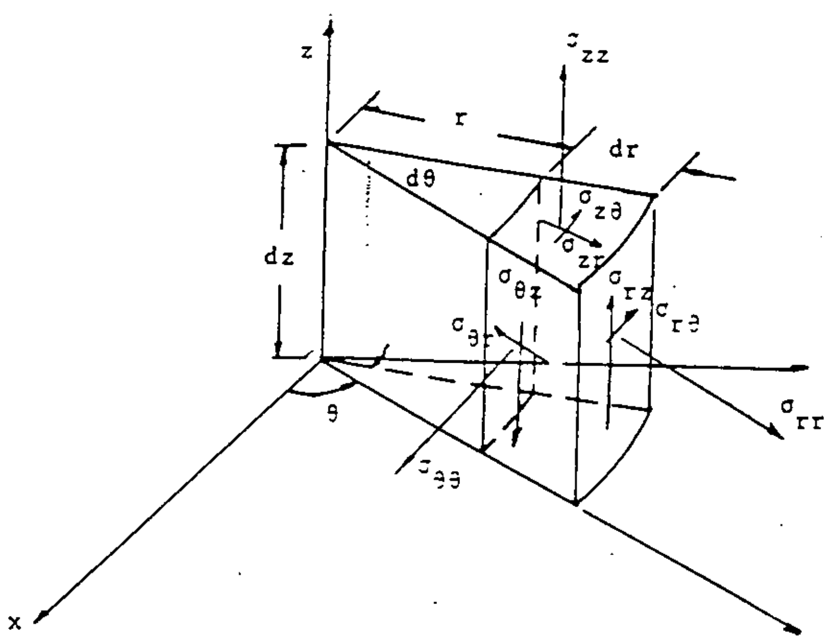

Fig.1.3-2 Stress tensor σ on a material element oriented in (r,θ,z) co-ordinates at a point

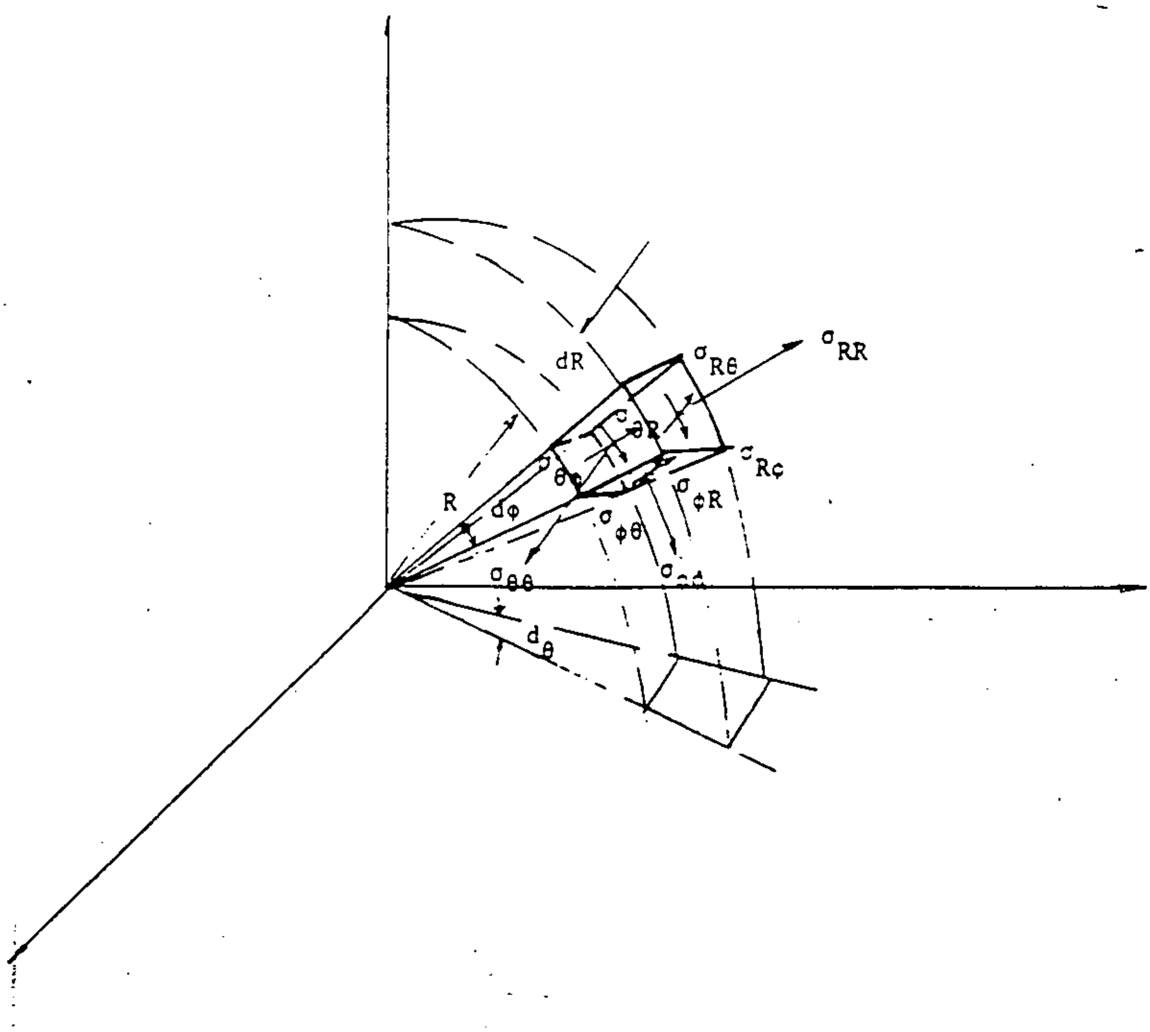

Fig.1.3-3 Stress tensor $\tilde{\sigma}$ on a material element oriented in (R,θ,ϕ) coordinates at a point

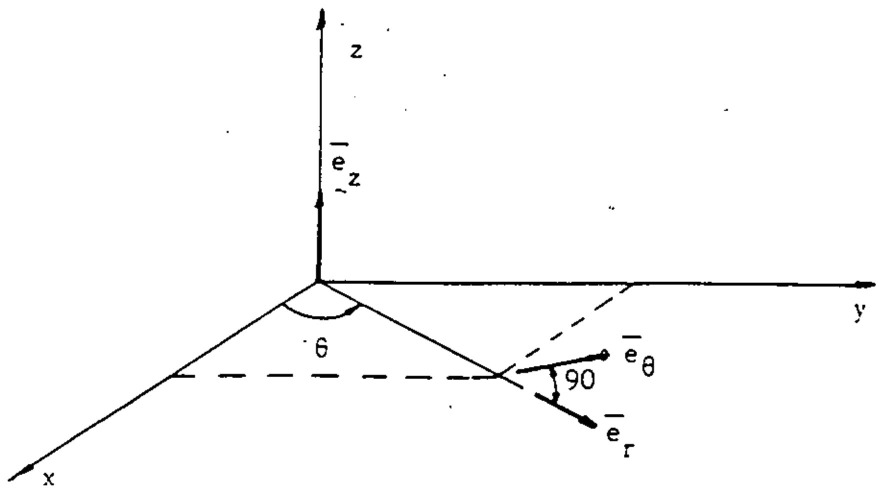

Fig.1.6-1 Unit vectors in cylindrical coordinate system (r,θ,z)

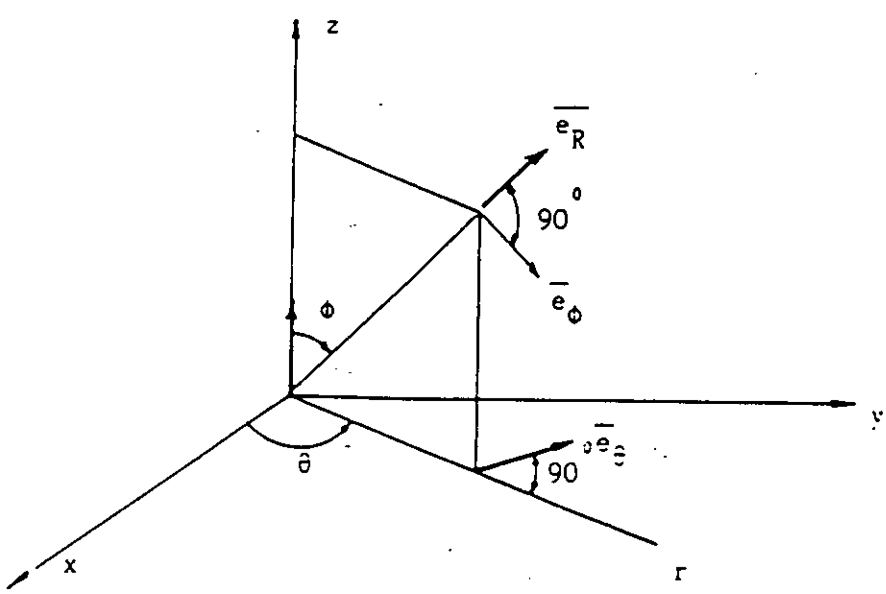

Fig.1.6-2 Unit vectors in spherical co-ordinate system (R,θ,ϕ)

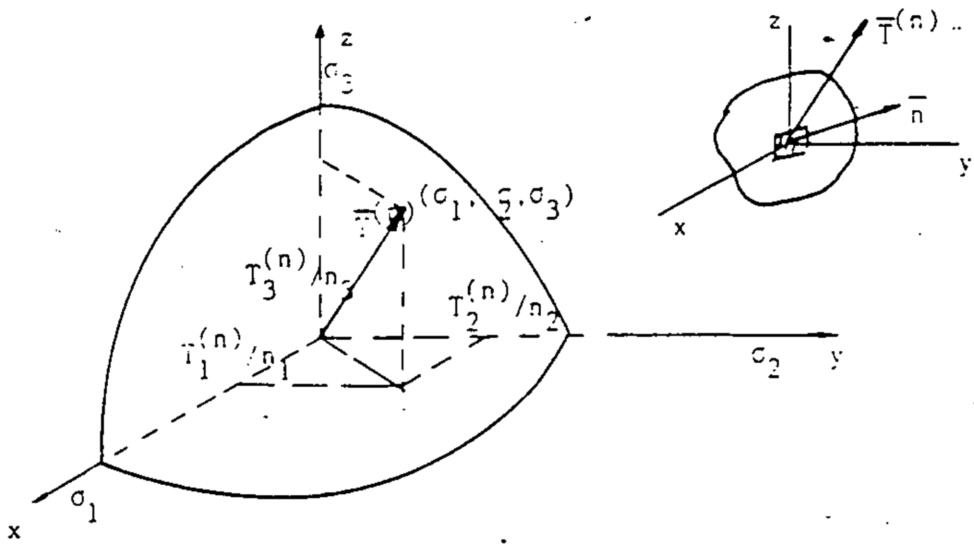

Fig.1.7-1 Lame' stress ellipsoid

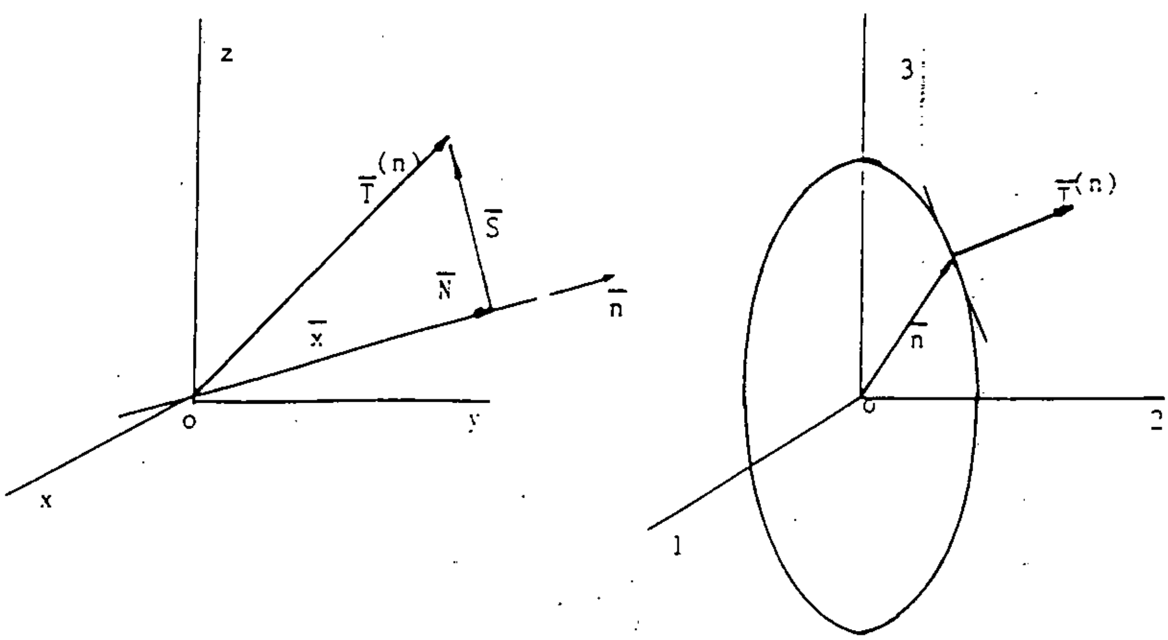

Fig.1.7-2 Cauchy's stress quatric

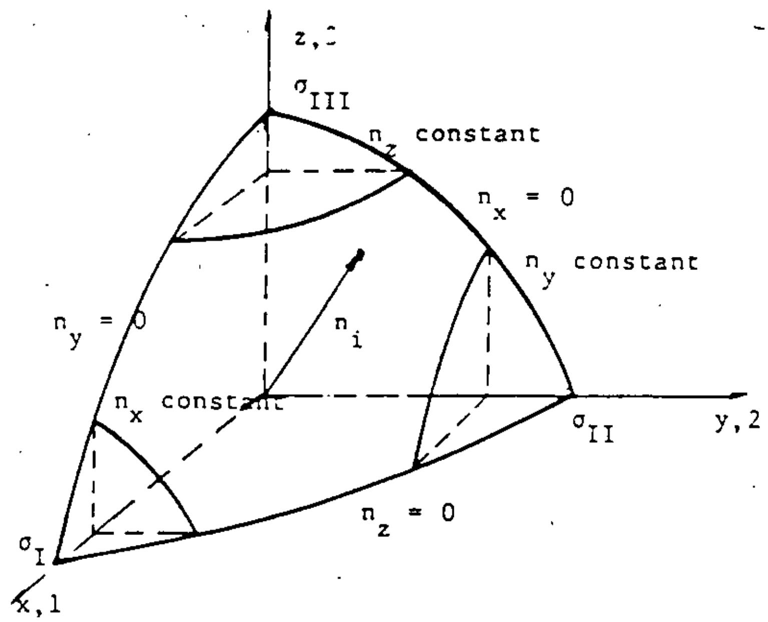

Fig.1.7-3 Mohr's circles on stress

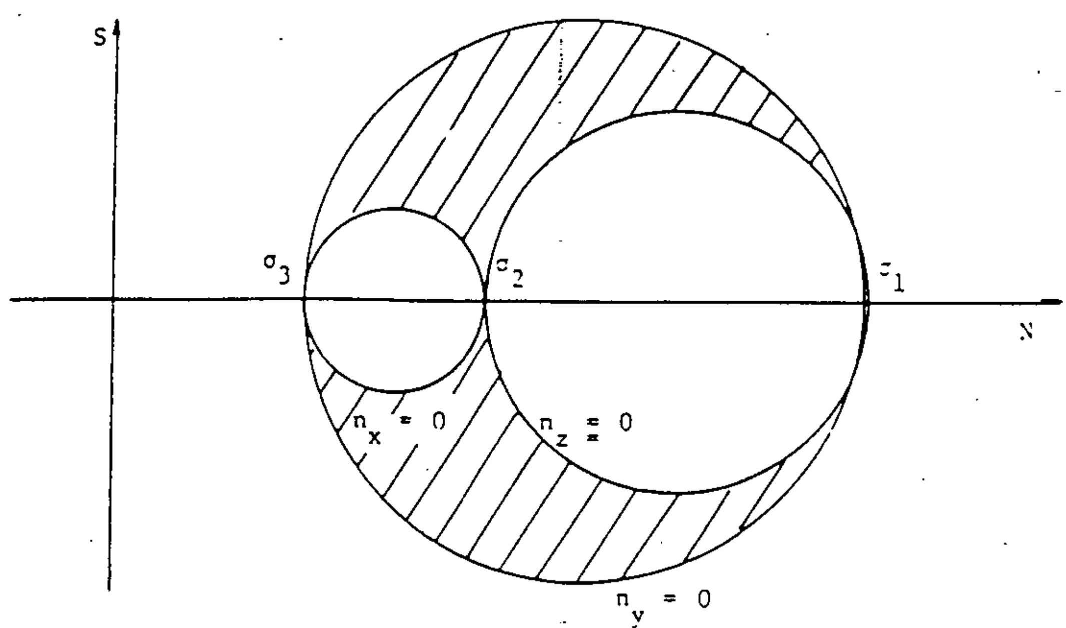

Fig.1.7-4 Mohr's circles on stress: absolute maximum shear

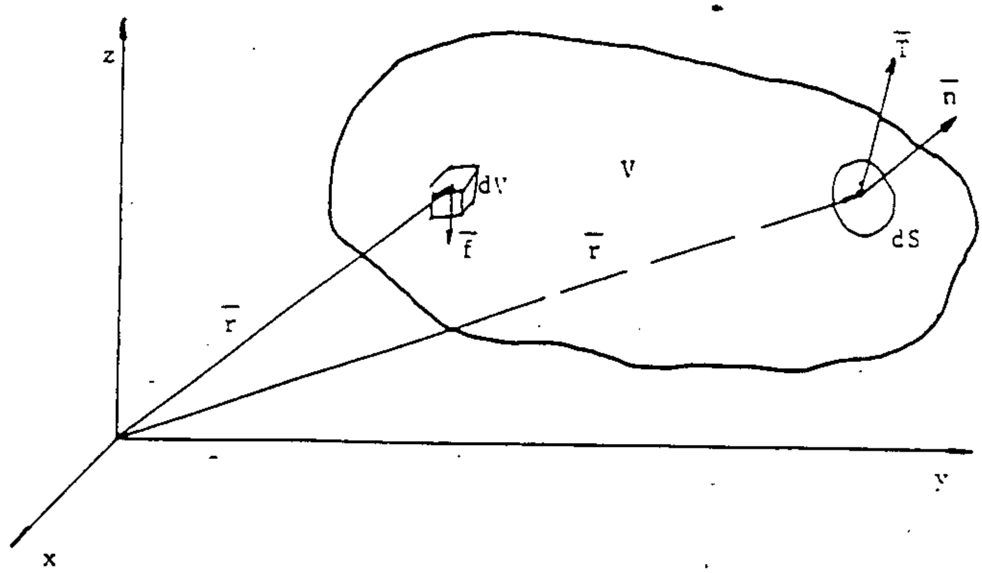

Fig.1.8-1 Solid body under surface traction $\overline{T}$ and body force $\overline{f}$

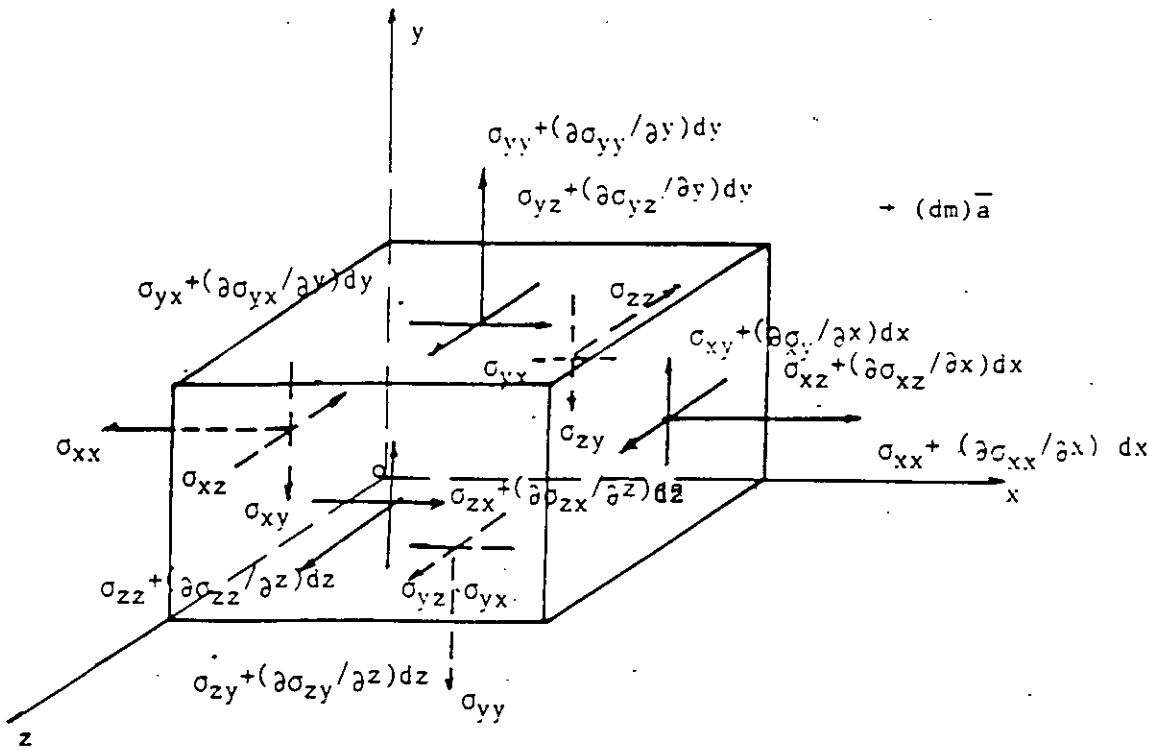

Fig.1.8-2 Equilibrium in the neighborhood of a point

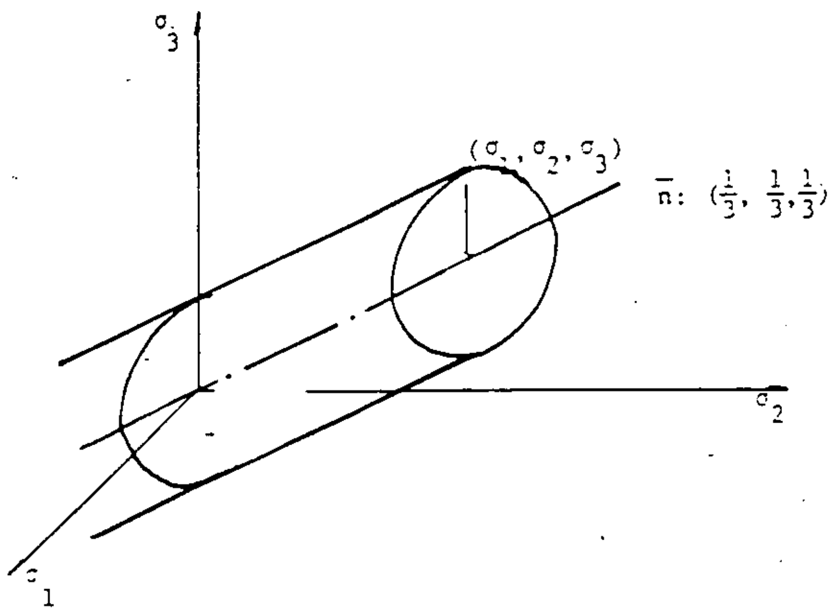

Fig.1.9-1

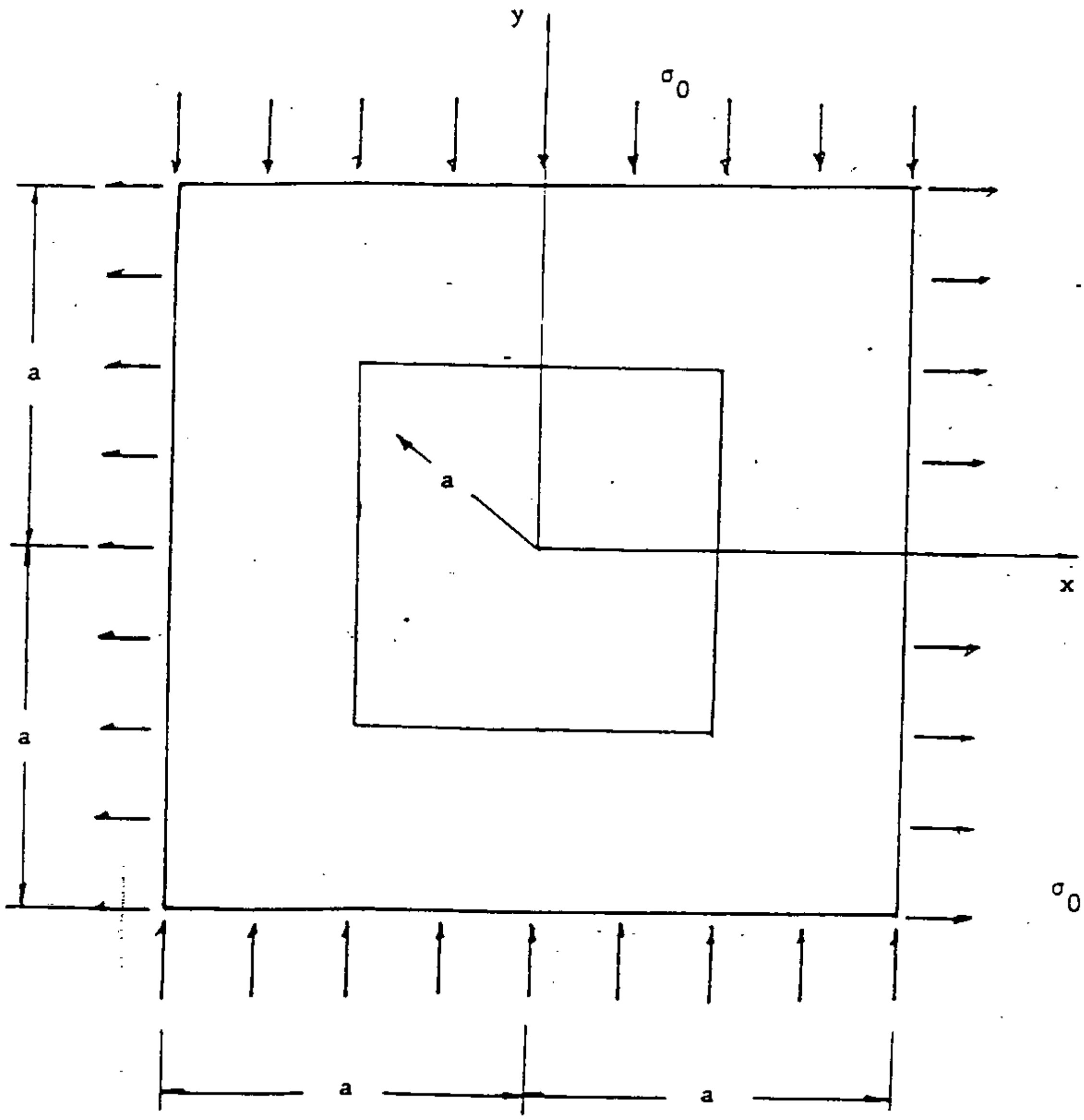

Fig.1.9-2 Square plate with center hole

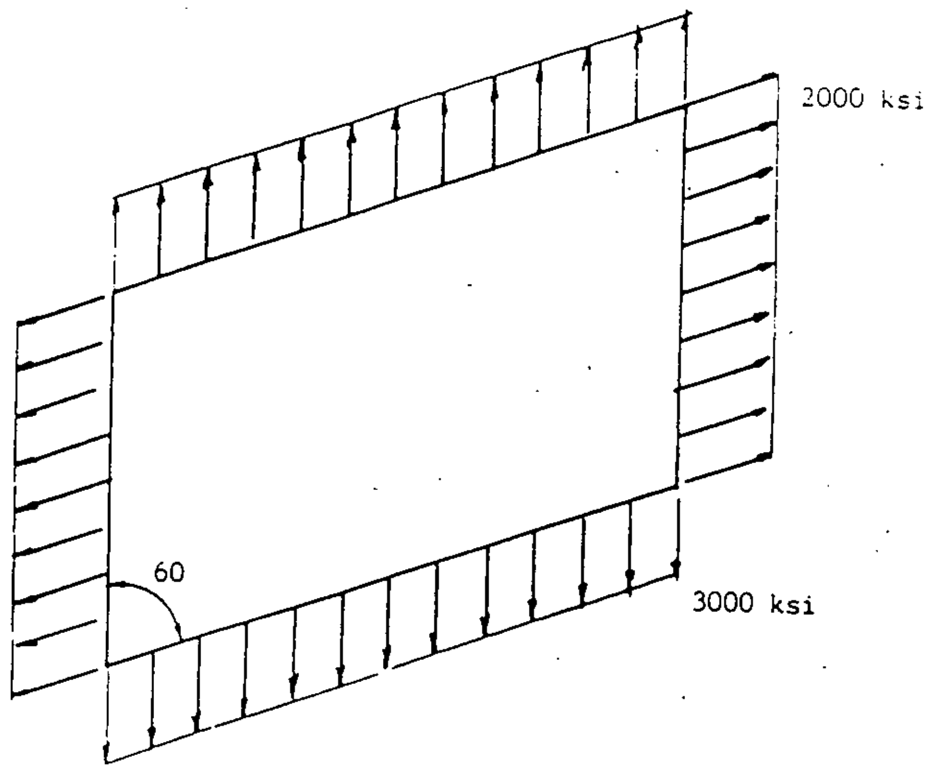

Fig.1.9-3 Thin plate under applied stresses