*X. VARIATIONAL PRINCIPLES OF MECHANICS

10.1 WORK AND ENERGY.

Unstrained or Natural State, Fig.10-1-1

A reference state of uniform temperature and zero displacement is called an unstrained or natural state.

Work Done and Rate of Work Done

work done $= W = \int_V f_i u_i dV + \int_{S_\sigma} T_i u_i dS$ (10.1-1)

rate of work done $= \dot{W} = \int_V f_i \dot{u}_i dV + \int_{S_\sigma} T_i u_i dS$ (10.1-2)

Work and Energy Principle

$= K + U$ (10.1-3)

K = kinetic energy

$= (1/2) \int \rho \dot{u}_i \dot{u}_i \, dV$ (10.1-4)

U = total strain energy

$= \int w \, dV$ (10.1-5)

w = volume density of strain energy density

$= w(\epsilon)$

$= C_0 + C_{ij} \epsilon_{ij} + (1/2) C_{ijkl} \epsilon_{ij} \epsilon_{kl}$

+ Higher Order Terms (10.1-6)

Here, K = 0 for elastic equilibrium state.

Generalized Hooke's Law

A generalized stress-strain relation in elasticity can be stated as:

$$\sigma_{ij} = \partial w(\epsilon)/\partial \epsilon_{ij} \qquad (10.1\text{-}7)$$

From Eqs.(10.2-6) and (10.2-7):

$$\sigma_{ij} = C_{ij} + C_{ijkl}\epsilon_{kl} + \text{H.O.T.} \qquad (10.1\text{-}8)$$

If stresses vanish with vanishing C_{ij}, then

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} + \text{H.O.T.} \qquad (10.1\text{-}9)$$

When higher order terms in Eq.(10.1-9) are dropped, we have the stress-strain relation for linear elasticity and the strain energy density can be written as

$$w = (1/2)\ \sigma_{ij}\epsilon_{ij} \qquad (10.1\text{-}10)$$

or $$w = w(\sigma) = (1/2)\ C^{-1}_{ijkl}\ \sigma_{ij}\ \epsilon_{kl} \qquad (10.1\text{-}11)$$

where $$C^{-1}_{ijkl}\ \sigma_{kl} = \epsilon_{kl} \qquad (10.2\text{-}12)$$

For isotropic elastic medium:

$$\sigma_{ij} = \lambda\sigma_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \qquad (10.1\text{-}13)$$

and $$w = w(\epsilon) = (1/2)\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij} \qquad (10.1\text{-}14)$$

or $$w = w(\sigma) = -(\nu/2E)\sigma_{ii}\sigma_{jj} + [(1+\nu)/2E]\sigma_{ij}\sigma_{ij} \qquad (10.1\text{-}15)$$

The strain energy density is therefore observed from Eq.(10.1-14) to be positive definite since the elasticity constants λ and μ take only positive values. This is not necessarily true when higher order terms in Eq.(10.1-6) are included.

<u>Uniqueness of Solution in Elasticity</u>

(1) Clapeyron's theorem: An elastic body under given applied body forces $\bar{f}$ and tractions $\bar{T}_i$ possesses an amount of strain energy that is equal to one-half the work done by the applied external forces acting through the displacement $\bar{u}$ from the unstrained state of equilibrium.

$$\int_V f_i u_i dV + \int_{S_\sigma} T_i^{(n)} u_i \, dS = 2\int wdV \qquad (10.1\text{-}16)$$

(2) Uniqueness of solution: Let $u_i^{(1)}$, $\sigma_{ij}^{(1)}$ and $u_i^{(2)}$, $\sigma_{ij}^{(2)}$ be two sets of solutions each satisfying the equations of elasticity,

Eqs.(4.1-1) to (4.1-7). It is clear that the difference between these solutions, u_i, σ_{ij} , where

$$u_i = u_i^{(1)} - u_i^{(2)}, \quad \sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)} \tag{10.1-17}$$

must satisfy

$$\sigma_{ij,i} = 0 \quad \text{and} \quad T_i^{(n)} = \sigma_{ji} n_j \quad \text{on } S_\sigma$$

Equation (10.1-16) thus reduces to

$$\int W dV = 0 \quad \text{or} \quad W = W(\epsilon) = 0$$

Since $W(\epsilon)$ is positive definite in strain, we must have

$$\epsilon_{ij} = \epsilon_{ij}^{(1)} - \epsilon_{ij}^{(2)} = 0$$

Hence the two solutions of stresses and displacements are identical:

$$\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)}, \quad u_i^{(1)} = u_i^{(2)}$$

Principle of Virtual Work

Let $\delta \bar{u}$ be the virtual displacement field and $\delta \epsilon$ the corresponding strain field, then the virtual work done by the external loads is:

$$\iiint f_i \delta u_i \, dV + \iint_{S_\sigma} T_i \, \delta u_i \, dS$$

where $\delta u_i = 0$ on S_u. Applying Gauss' theorem, Eq.(A.1-24) and using the equations of equilibrium, Eqs.(1.8-2), it can easily be shown that

$$\iiint_V f_i \delta u_i dV + \iint_{S_\sigma} T_i \delta u_i dS = \iiint_V \sigma_{ij} \delta \epsilon_{ij} dV \qquad (10.1\text{-}18)$$

where

$$\sigma_{ij} = \partial w(\epsilon)/\partial \epsilon_{ij} \qquad (10.1\text{-}7)$$

The principle of virtual work states that the virtual work done by the externally applied traction and body forces is equivalent to the virtual work done by the internal stresses. The validity of Eq.(10.1-18) is independent of the material behavior, linear or non-linear.

Principle of Virtual Complementary Work

Consider a system of stress in equilibrium. Let $\delta \bar{f}$ and $\delta \bar{T}$ be the body force and traction, respectively, and $\delta \bar{\sigma}$ be the corresponding equilibrium stress field. The complementary virtual work,

$$\iiint_V u_i \delta f_i \, dV + \iint_{S_\sigma} u_i \delta T_i \, dS$$

can be easily shown, through integration by parts, to be equivalent to the total complementary strain energy

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$$\iiint_V u_i \delta f_i \, dV + \iint_S u_i \, \delta T_i \, dS = \iiint_V \epsilon_{ij} \delta\sigma_{ij} \, dV \qquad (10.1\text{-}19)$$

in which

$$\epsilon_{ij} = \partial\bar{w}(\sigma)/\partial\sigma_{ij}$$

and $\delta\sigma_{ij}$ is arbitrary on S_u.

Reciprocal Theorem of Betti and Rayleigh, Fig.10.1-2

If an elastic body is subjected to two system of body and surface forces $\bar{f}$, $\bar{T}$ and $\bar{f}'$, $\bar{T}'$, then the work that would be done by the first system in acting through the displacements $\bar{u}'$ due to the second system of forces is equal to the work that would be done by the second system in acting through the displacements $\bar{u}$ due to the first system, i.e.

$$\iint_S u_i' T_i \, dS + \iiint_V u_i' f_i \, dV = \iint_S u_i T_i' \, dS + \iiint_V u_i f_i' \, dV \qquad (10.1\text{-}20)$$

The above equation can be written in terms of stresses and strain as follows:

$$\iiint_v \sigma_{ij} \epsilon_{ij}' \, dV = \iiint s_{ij}' \epsilon_{ij} \, dV \qquad (10.1\text{-}21)$$

Reciprocal Theorem: Maxwell

Let $\qquad u_i = G_{ij} \, T_j$

$$u_i' = G_{ij}\, T_j'$$

then it is clear that

$$G_{ij} = G_{ji} \qquad (10.1\text{-}22)$$

Theorems of Castigliano

(1) Let $U = U(\bar{T}_i)$

then $\partial U/\partial T_i = u_i, \quad i = 1,2,\cdots,n. \qquad (10.1\text{-}23)$

(2) Let $U = U(\bar{u})$

then $\partial U/\partial u_i = T_i, \quad i = 1,2,\cdots,n. \qquad (10.1\text{-}24)$

10.2 VARIATIONAL PRINCIPLES OF MECHANICS

Principle of Minimum Potential Energy

The potential energy π, where

$$\Pi = \iiint_V W\, dV - \iint_S u_i T_i\, dS - \iiint_V u_i f_i\, dV \qquad (10.2\text{-}1)$$

assumes a minimum value when the displacement field $\bar{u}$ is that of the equilibrium state, i.e.

$$\delta\Pi = 0 \qquad (10.2\text{-}2)$$

In other words, of all displacement fields that satisfy the given boundary conditions that which satisfy the equilibrium conditions makes the potential energy an absolute minimum.

Principle of Minimum Complimentary Energy

The complimentary energy Π^*, where

$$\Pi^* = \int w_c(\bar{\sigma})\, dV - \int_V u_i f_i\, dV - \int_{S_u} u_i T_i\, dS \tag{10.2-3}$$

assumes a minimum value when the stress field σ is that gives rise to compatible strain field, i.e.

$$\delta\Pi^* = 0 \tag{10.2-4}$$

In other words, of all stress fields that satisfy the equations of equilibrium and boundary conditions where stresses are prescribed, the "actual" one is distinguished by a stationary value of the complimentary energy.

Reissner's Principle

(1) Potential energy - linear elasticity

Let a functional J be minimized, i.e.

$$\delta J = 0 \tag{10.2-5}$$

where $$J[\bar{\epsilon},\ \bar{u},\ \bar{\sigma}] = \int_V [W(\bar{\epsilon}) - u_i f_i] dV$$

$$- \int_V \sigma_{ij}[\epsilon_{ij} - (u_{i,j} + u_{j,i})/2] dV$$

$$- \int_{S_\sigma} T^\star u_i \, dS - \int_{S_u} \sigma_{ji} n_j (u_i - u_i^\star) \, dS \qquad (10.2\text{-}6)$$

in which $\sigma_{ij} = \sigma_{ji}$, $S = S_\sigma + S_u$.

The Euler equations can be easily shown to be:

$$\partial W/\partial \epsilon_{ij} = \sigma_{ij} \quad \text{in } V \qquad (10.2\text{-}7)$$

$$\sigma_{ji,j} + f_i = 0 \quad \text{in } V \qquad (10.2\text{-}8)$$

$$\epsilon_{ij} = [u_{i,j} + u_{j,i}]/2 \quad \text{in } V \qquad (10.2\text{-}9)$$

$$\sigma_{ij} n_j = T_i^\star \quad \text{on } S_\sigma \qquad (10.2\text{-}10)$$

$$u_i = u_i^\star \quad \text{on } S_u \qquad (10.2\text{-}11)$$

(2) Complementary energy

$$W_c(\sigma_{ij}) = \sigma_{ij}\epsilon_{ij} - W(\epsilon_{ij}) \qquad (10.2\text{-}12)$$

$$\partial W_c/\partial \sigma_{ij} = \epsilon_{ij} \qquad (10.2\text{-}13)$$

$$J_R = \int_V(-W_c(\sigma_{ij}) - F_i u_i + \sigma_{ij}(u_{i,j} + u_{j,i})/2)dV$$

$$-\int_{S_\sigma} T_i^{\star} u_i \; dS - \int_{S_u} \sigma_{ij} n_j (u_i - u_i^{\star}) \qquad (10.2\text{-}14)$$

10.3 PROBLEMS ON VARIATIONAL CALCULUS

1. Write down the appropriate Euler equation(s) for

   (a) A two dependent and one independent variable functional,

   (b) $F = m(x\, z_x^2 + L\, z_y^2)/2 + kz$, where $z = z(x,y)$ and m, L, k are constants.

Ans. (a) If $I = \int_{x_1}^{x_2} F(x,u,v,u_x,v_x)\, dx$ and $\delta I = 0$

then the appropriate Euler equations are

$$\partial F/\partial u - d(\partial F/\partial u_x)dx = 0$$

$$\partial F/\partial v - d(\partial F/\partial v_x)dx = 0$$

(b) Appropriate Euler equation is

$$\partial F/\partial z - \partial(\partial F/\partial z_x)/\partial x - \partial(\partial F/\partial z_y)\partial y = 0$$

hence $x\, \partial^2 z/\partial x^2 + L\, \partial^2 z/\partial y^2 + z_x = k/m.$

2. Of all rectangular parallelpipes which have sides parallel to the co-ordinate planes, and which are inscribed in the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

determine the dimensions of that one which has the largest possible volume.

Ans. $f = 8xyz$, $\phi = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1$

$$yz + \lambda_1 (2x/a^2) = 0$$

$$xz + \lambda_1 (2y/b^2) = 0$$

$$xy + \lambda_1 (2z/c^2) = 0$$

$$\rightarrow \lambda_1 = -yza^2/2x, \quad x = \pm a/\sqrt{3}, \quad y = \pm b/\sqrt{3}, \quad z = \pm c/\sqrt{3}$$

$$f = 8abc/(9\sqrt{3})$$

3. Obtain Euler equation(s) and natural boundary condition(s) for stationary values of

$$\int_0^1 F(x,y,y')\,dx$$

(a) $F = y'^2 + yy' + y^2$

(b) $F = y'^2 + k^2 \cos y$

Ans. (a) Euler equation: $y'' - y = 0$

natural boundary condition: $y(0) = 1, \quad y(1) = e^{-1/2}$

(b) Euler equation: $2y'' + k^2 \sin y = 0$

natural boundary condition: $y =$ constant at $x = 0, \ x = 1$

4. Solve by the Rayleigh-Ritz procedure

$$d(x\, dy/dx)/dx + y = x, \quad y(0) = 0, \ y(1) = 1$$

with $\bar{\bar{y}} = x + x(1 - x)(c_1 + c_2 x)$

Ans. The trial function $\bar{\bar{y}}$ satisfies boundary conditions and is therefore admissible. The equivalent problem is

$$\int_0^1 [(xy')' + y - x]\ \delta y\ dx = 0$$

where $\delta y = (x - x^2)\ \delta c_1 + (x^2 - x^3)\ \delta c_2$

Integrating between limits yields: $c_1 = 85/26, \quad c_2 = -35/13.$

10.4 BENDING OF A BEAM

Let the distributed loads w(x), shear V, moment M be applied in one of the principal plane and the principal axes of inertia of every cross section of the beam lie in two mutually orthogonal principal planes. All plane cross sections are assumed to remain plane during bending.

The potential energy is

$$\Pi = \frac{1}{2} \int_0^L EI(y'')^2 \, dx - \int_0^L w(x) \, y dx + M_0 y'(0) - M_1 y'(x_1)$$

$$-V_0 y(0) + V_1 y(x_1)$$

and the condition that

$$\delta\Pi = 0$$

with respect to virtual displacement δy leads to

$$\int_0^L [d^2(EIy'')/dx^2 - p] \, \delta y \, dx + [EIy''(L) - M_1]\delta y'(L)$$

$$-[EIy''(0) - M_0]\delta y'(0) - (d[EIy''(L)]/dx - V_1]\delta y(L)$$

$$+(d[EIy''(0)]/dx - V_0)\delta y(0) = 0$$

From which the Euler equation is,

$$d^2(EIy'')/dx^2 - w = 0$$

the natural boundary conditions are

$$EIy''(L) - M_L = 0$$

$$EIy''(0) - M_0 = 0$$

$$d[EIy''(L)]/dx - V_L = 0$$

$$d[EIy''(0)]/dx - V_0 = 0$$

and the rigid boundary conditions are

$$\delta y'(L) = 0,\ \delta y'(0) = 0,\ \delta y(L) = 0,\ \delta y(0) = 0$$

Discuss the admissibility of trial functions in using the direct methods of variational problems.

Straight Beam With Uniformly Distributed Load, Fig.10.4-1

Let $$y(x) = \sum_{n=1}^{\infty} a_n \sin (n\pi x/L)$$

it is clear that y(x) satisfy rigid boundary conditions at $x = 0$ and $x = L$

$$\Pi(a_n) = \int_0^L (EI/2)(d^2y/x^2)^2 dx - \int_0^L wy\, dx$$

$$= [(EI\pi^4)/(4L^3)] \sum_{n=1}^{\infty} n^4 a_n^2 - (2w_0 L/2) \sum_{n=1,3,5\cdots}^{\infty} a_n/n$$

For an extremum value of π, let

$$\partial\Pi/\partial a_n = 0$$

from which

$$a_n = (4w_0L^4)/(EI\pi^5 n^5) \qquad n = 1,3,5,\cdots$$

$$a_n = 0 \qquad n = 0,2,4,\cdots$$

Hence

$$y(x) = [4w_0L^4/(EI\pi^5)] \sum_{n=1,3,5,\cdots}^{\infty} (1/n^5) \sin n\pi x/L$$

At $x = L/2$,

$$y(L/2) = [4w_0L^4/(EI\pi^5](1 - 1/3^5 + 1/5^5 - \cdots)$$

$$= w_0L^4/(76.8EI)$$

Compute the result for y(L/2) by taking only the first term and compare it with the exact answer.

10.5 TORSION OF A PRISMATIC BAR, Fig.10.5-1

Let

$$\sigma_{xz} = \partial\phi/\partial y, \quad \sigma_{yz} = - \partial\phi/\partial x$$

then the total complementary energy is

$$\Pi^\star = [L/(2\mu)] \iint_A [(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 - 4\mu\alpha\phi]dA$$

$$+ \alpha L \oint_c \phi\ (xdy - ydx)$$

The condition $\delta\Pi^* = 0$ yields

$$\delta\Pi^* = (L/2\mu) \iint_A [2\partial\phi/\partial y \; \partial\delta\phi/\partial y + 2\partial\phi/\partial x \; \partial\delta\phi/\partial x$$

$$-4\mu\alpha(\delta\phi)] \; dA + \oint_C \alpha L \; \delta\phi(xdy - ydx) = 0$$

Hence the Euler equation is

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = -2\mu\alpha \qquad \text{in } A$$

the boundary condition is

$$\delta\phi = 0 \qquad \text{on } C$$

or ϕ is equal to a constant which can be taken to be zero.

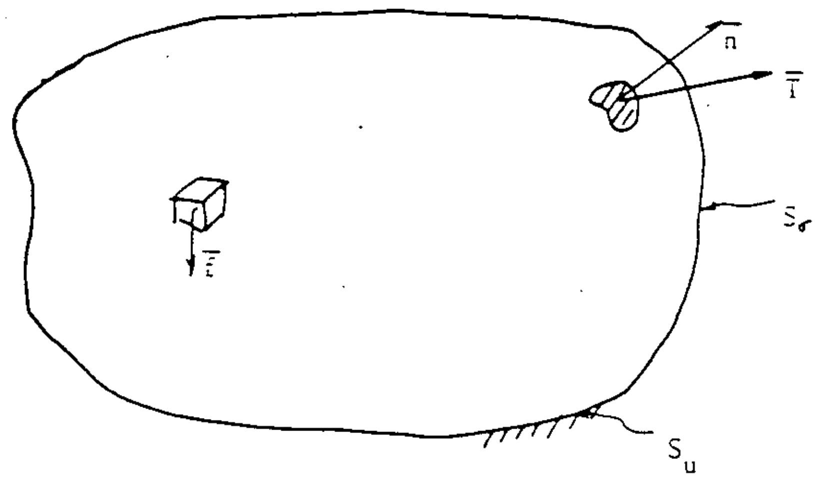

Fig.10.1-1

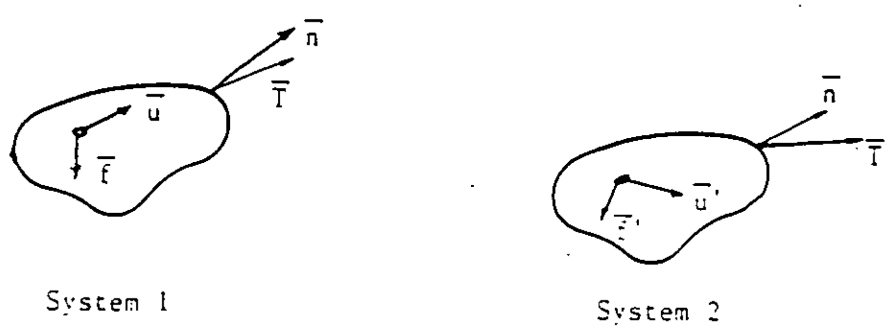

Fig.10.1-2 Reciprocal theorem

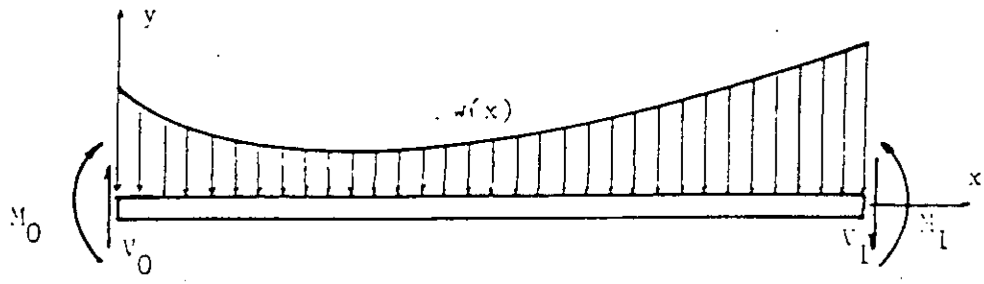

Fig.10.3-1

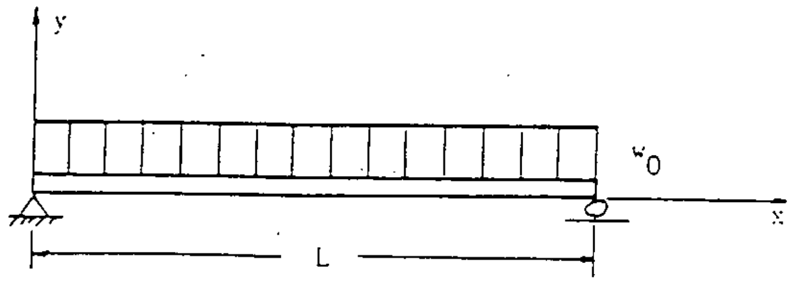

Fig. 10.3-2 Distributed load $w(x) = w_0$, $0 \leq x \leq L$
$y(0) = 0$, $y(L) = 0$

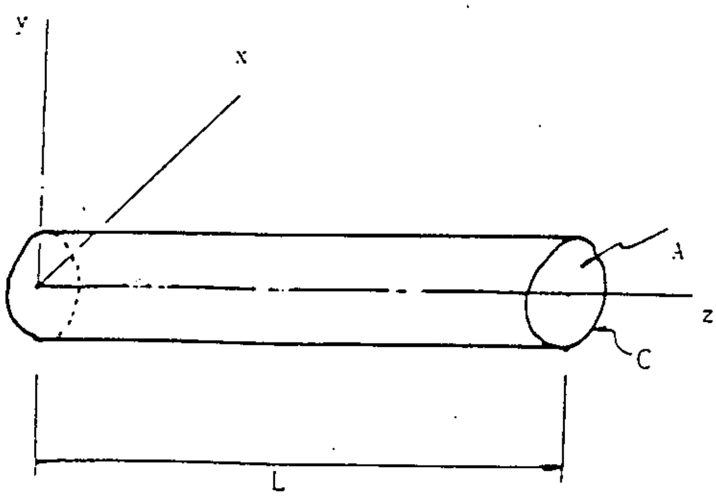

Fig. 10.4-1

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