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Problem 1

Consider an infinitely long column of liquid of density ρ, radius a, and interfacial surface tension γ rotating at uniform angular velocity Ω. The inviscid incompressible ($\rho = const.$) motion of the Euler equations in a system rotating at angular velocity Ω in dimensional (asterisk) coordinates are given by

$$\frac{D\mathbf{u}^*}{Dt} + 2\mathbf{\Omega} \times \mathbf{u}^* = -\frac{1}{\rho}\nabla^* p^* + \nabla^* \left(\frac{|\mathbf{\Omega} \times \mathbf{r}^*|^2}{2} \right)$$

$$\nabla^* \cdot \mathbf{u}^* = 0$$

where we take $\mathbf{\Omega} = \Omega\mathbf{k}$ for rotation about the z^* axis and $\mathbf{r}^*$ is the position vector. Show for cylindrical coordinates (r^*, θ, z^*) with associated velocities (u^*, v^*, w^*) that the base flow is

$$\left.\begin{aligned} u^* &= v^* = w^* = 0 \\ p^* &= p_0^* + \frac{1}{2}\rho\Omega^2 r^{*2} \end{aligned}\right\} \qquad (0 \le r^* \le a).$$

where p_0^* is the pressure at $r^* = 0$. Assuming $p^* = p_\infty^*$ in the gas surrounding the liquid column ($r^* \ge a$), show that the kinematic and dynamic free surface boundary conditions are

$$\left.\begin{aligned} u^* &= \eta_{t^*}^* + \frac{v^*}{r^*}\eta_\theta^* + w^*\eta_{z^*}^* \\ p &= p_\infty^* + \gamma\nabla^* \cdot \mathbf{n} \end{aligned}\right\} \qquad (r^* = a + \eta^*).$$

where η^* is the disturbed position of the free surface about $r^* = a$ and $\mathbf{n}$ is the outward normal to the interface. Normalizing disturbances by a, time by Ω^{-1}, velocities by Ωa, and pressure by $\rho\Omega^2 a^2$, show that the dimensionless equations are

$$\begin{aligned} \frac{Du}{Dt} - \frac{v^2}{r} - 2v &= -\frac{\partial}{\partial r}\left(p - \frac{r^2}{2}\right) \\ \frac{Dv}{Dt} + \frac{uv}{r} + 2u &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} \\ \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} &= 0 \end{aligned}$$

Part A

Now investigate the temporal stability of the flow by positing disturbances of the form

$$\begin{pmatrix} u \\ v \\ w \\ p \\ \eta \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \\ P \\ \eta_0 \end{pmatrix} e^{i(kz+n\theta)+st}$$

wherein $s = \sigma + i\omega$ and $U(r), V(r), W(r)$ and$P(r)$ are radial eigenfunctions. Note that $n = 0$ for $k \neq 0$ describes axisymmetric disturbances; $k = 0$ for $n \neq 0$ describes planar disturbances; and $n \neq 0$ with $k \neq 0$ descirbes nonaxisymmetric (spiral) disturbances ask sketched in Figure 1. Show that

$$U = -\frac{1}{s^2+4}\left[sP_r + \frac{2in}{r}P\right]$$

$$V = \frac{1}{s^2+4}\left[2P_r - \frac{ins}{r}P\right]$$

$$W = -\frac{ik}{s}P$$

and

$$P_{rr} + \frac{1}{r}P_r - \left[\frac{n^2}{r^2} + \beta^2\right]P = 0 \qquad \text{where} \quad \beta^2 = \frac{k^2(s^2+4)}{s^2}$$

with solution finite at $r = 0$ given by

$$P(r) = AI_n(\beta r).$$

Finally, use the kinematic and dynamic boundary conditions to obtain the eigenvalue equation

$$\beta\frac{I_n'(\beta)}{I_n(\beta)} = \frac{s^2+4}{1+(1-k^2-n^2)L} - \frac{2in}{s} \tag{1}$$

where primes denote differentiation with respect to r. Hence show that for axisymmetric disturbances one finds

$$\beta\frac{I_n'(\beta)}{I_n(\beta)} = -\frac{s^2+4}{\psi_1} \qquad \text{where} \quad \psi_1 = -1 + L(k^2-1) \tag{2}$$

Verify that in the limit $\Omega \to 0$ one recovers the eigenvalue equation found by Rayleigh (1879) for axisymmetric disturbances of a nonrotating fluid column. (note that in the nondimensionalization s^* is normalized by Ω.

with boundary conditions

$$\left.\begin{aligned} u &= \eta_t + \frac{v}{r}\eta_\theta + w\eta_z \\ p - p_\infty &= L\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{D}\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\eta_\theta}{D}\right) - \frac{\partial}{\partial z}\left(\frac{\eta_z}{D}\right)\right] \end{aligned}\right\} \qquad (r = 1+\eta).$$

where

$$D = \sqrt{1 + \frac{1}{r^2}\eta_\theta^2 + \eta_z^2}$$

and the base flow is $\mathbf{u} = 0$, $p = p_0 + r^2/2$ for $0 \le r \le 1$.

Investigate stability to small disturbances

$$\begin{aligned} u &= \epsilon u' & \eta &= \epsilon\eta' \\ v &= \epsilon v' & & \\ w &= \epsilon w' & p &= p_0 + \frac{r^2}{2} + \epsilon p' \end{aligned}$$

to obtain the linearized system (dropping primes)

$$\begin{aligned} \frac{\partial u}{\partial t} - 2v &= -\frac{\partial p}{\partial r} \\ \frac{\partial v}{\partial t} + 2u &= -\frac{1}{r}\frac{\partial p}{\partial\theta} \\ \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} \\ \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial z} &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} &p \quad \text{finite} && @r = 0 \\ &u = \eta_t && @r = 1 \\ &p = -[\eta + L(\eta + \eta_{\theta\theta} + \eta_{zz})] && @r = 1 \end{aligned}$$

where $L = \gamma/\rho\Omega^2 a^3$ is the parameter introduced by Hocking (*Mathematica*, 7, 1-9, 1960). Actually L is a rotating Weber number.

Part B

Show, without explicitly solving (2) for axisymmetric disturbances, that information about stability may be obtained by writing the $n = 0$ problem in the form

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dG}{dr}\right) - \frac{\beta^2}{r}G = 0 \tag{3a}$$

where $G(r) = rU(r)$ and the associated boundary condtions are

$$G(0) = 0 \tag{3b}$$

$$G_r(1) + \frac{k^2}{s^2}\psi_1 G(1) = 0. \tag{3c}$$

Derive the functional for s^2 by first multiplying (3a) by G, integrating over the domain $[0, 1]$ using integration by parts, and applying bounary conditions (3b,c). Hence show that

$$-\frac{s^2}{k^2} = \frac{4\int_0^1 \frac{G^2}{r}dr + \psi_1 G^2(1)}{\int_0^1 \left\{\frac{1}{r}[G_r^2 + k^2G^2]\right\} dr}$$

and show that this infers that axisymmetrically disturbed flow is stable only if $\psi_1 > 0$. Show also that instability is possible only for wavenumbers below a cutoff wavenumber k_0 given by

$$k_0 = \sqrt{1 + \frac{1}{L}}.$$

Growth rate curves for $n = 0$ computed (2) for $s = \sigma$ plotted in Figure 2 exhibit a common intersection at $k = 1$. Show using (2) that the crossover occurs at $\sigma = 0.43323$ and verify, using the numerical values in Figure 2 for at least one curve, that all unstable growth rates lie in the region $0 < k < k_0$.

A cross plot of the maximum growth rate σ_m and the associated critical wavenumbers k_c obtained from many numerical calculations at different values of L is given in Figure 3. Note that all k_c satisfy $k_c < k_0$ as must be the case. Comparison with the asymptotic results of Rayleigh (1879) for $L \to \infty$ and Pedley (1967) for $L \to 0$ are also shown in the figure.

Part C

Spiral and planar disturbances can compete for instability if their maximum growth rates exceed (for any ω) the values σ_m obtained in Figure 3. Numerical calculations show that $n \geq 1$ spiral disturbances have values of σ_m less than those for axisymmetric disturbances at each L. Planar disturbances, however, do compete for instability at sufficiently low L.

Demonstrate this result by showing first that planar disturbances ($k = \beta = 0$) are governed by the boundary value problem

$$
\begin{aligned}
r^2 P_{rr} + r P_r - n^2 P = 0 & \\
P = \frac{1 + L(1-n^2)}{s^2+4}\left(P_r + \frac{2in}{s}P\right) & \quad @ \quad r = 1 \\
P \quad \text{finite} & \quad @ \quad r = 0
\end{aligned}
$$

and show that the solution finite at $r = 0$ is

$$P(r) = Cr^n \qquad (n \geq 1).$$

Also show that solution of the eigenvalue relation gives

$$s = i \pm \sqrt{n\psi_2 - 1}$$

where now $\psi_2 = 1 + L(1 - n^2)$. Thus show that the flow is neutrally stable for $n = 1$ and unstable for

$$L < \frac{1}{n(n+1)} \qquad (n \geq 2).$$

Hence fluid disturbances rotating at $\omega^* = \Omega$ with respect to the rotating frame have growth rates $\sigma = \pm\sqrt{n\psi_2 - 1}$. Also show that $n = 2$ instability gives way to higher planar modes ($n = 3, 4, 5$ etc.) as $L \to 0$. Show that these transition points L_t between the unstable planar modes are given by

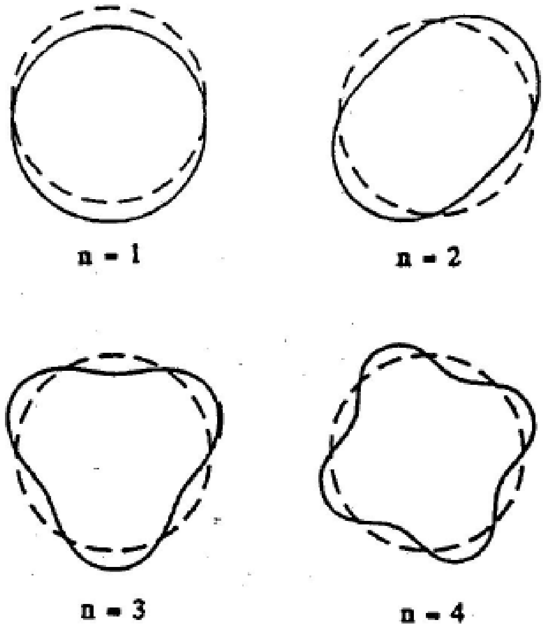
$$L_t = \frac{1}{3n(n+1)}.$$

Finally, use the preceeding results to show that the planar mode growth rates first exceed the maximum axisymmetric growth rates at $n = 2$. This is most easily done by calculating the crossover point between mode 2 grow rates with the ($L \to 0$) asymptotic results plotted in Figure 3; the equations describing the asymptotic curves σ_m and k_c are given by (Pedley, 1967), *viz.*

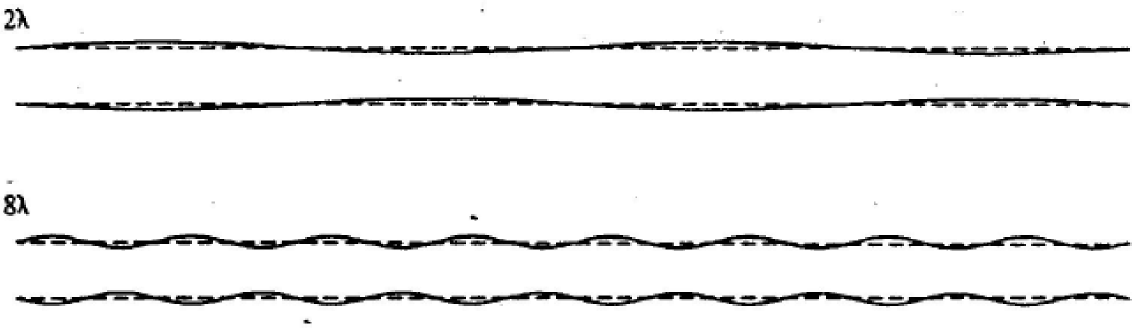
$$\sigma_m^2 = \frac{2(1+L)}{27L + [(1 + 30L + 3L^2 + L^3)]^{1/2}}$$

$$k_c^2 = \frac{1+L}{3L}.$$

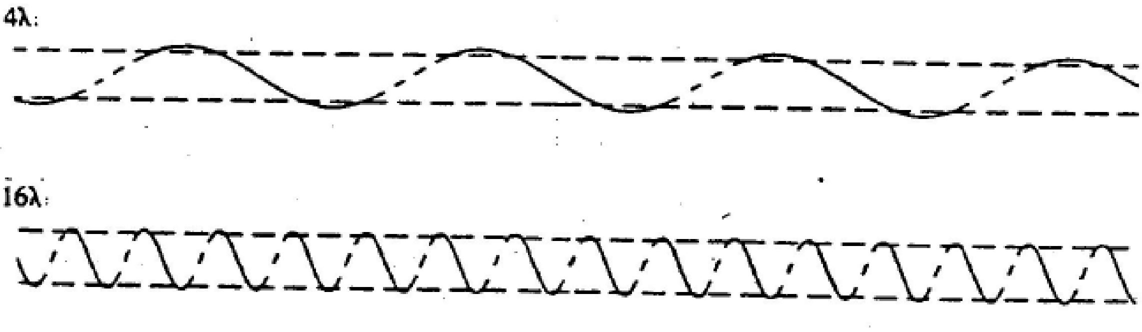
Find the approximate crossover value L_c and compare it with the numerically computed crossover value $L_c = 0.1053$ and hence show that the stability diagram at low values of L looks as shown in Figure 4.



(a) Planar modes: k = 0



(b) Axisymmetric modes: n = 0



(c) Spiral modes

FIG. 1

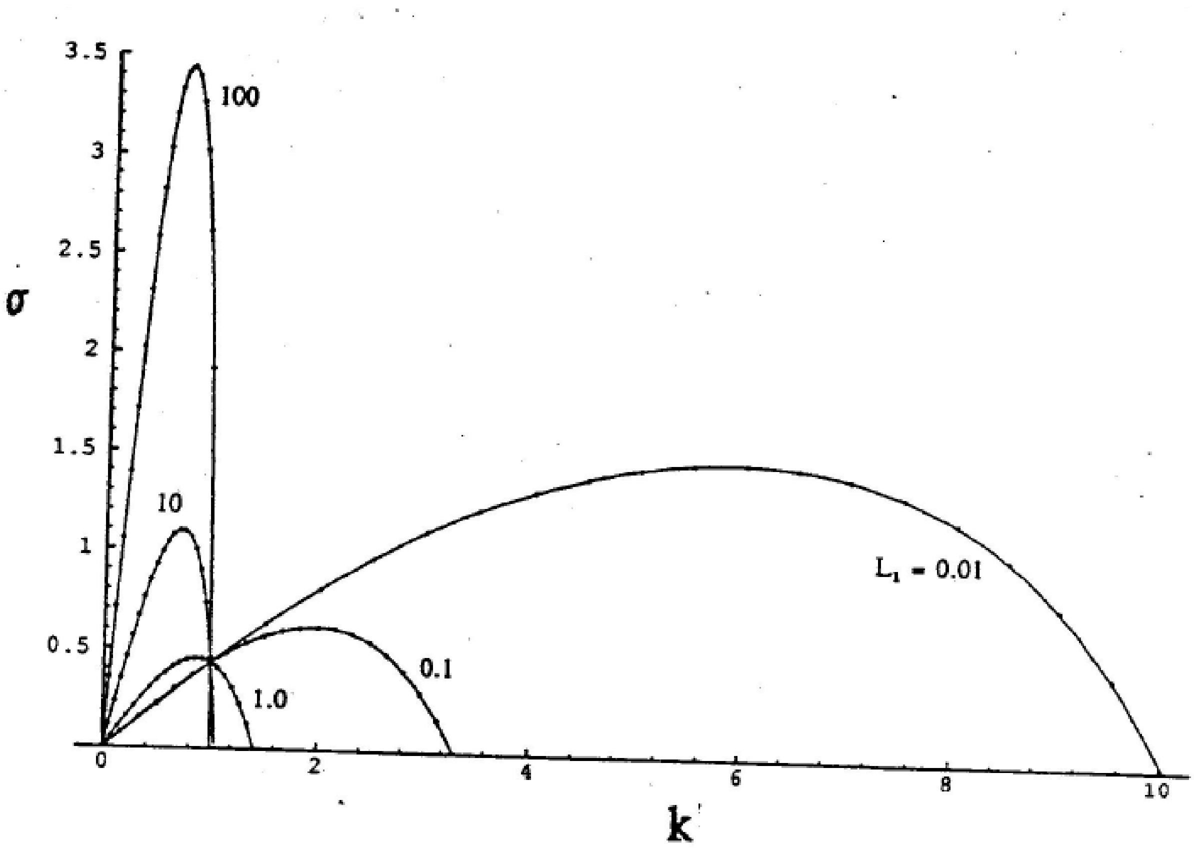


FIG. 2

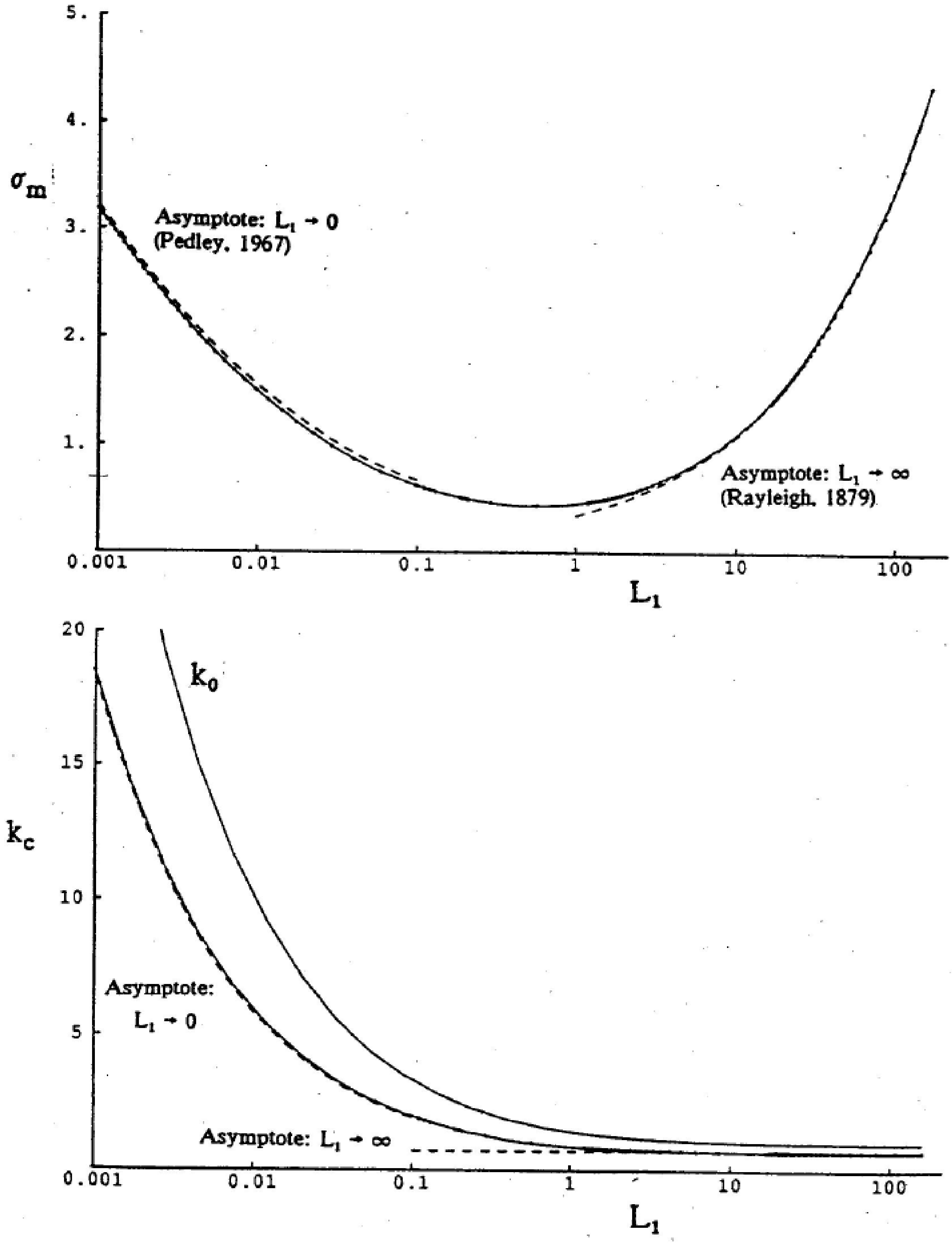

5.
4.
σ_m
3.
2.
1.
Asymptote: $L_1 \to 0$
(Pedley, 1967)
Asymptote: $L_1 \to \infty$
(Rayleigh, 1879)
0.001
0.01
0.1
1
10
100
L_1
20
15
10
5
k_c
k_0
Asymptote:
$L_1 \to 0$
Asymptote: $L_1 \to \infty$
0.001
0.01
0.1
1
10
100
L_1

FIG. 3

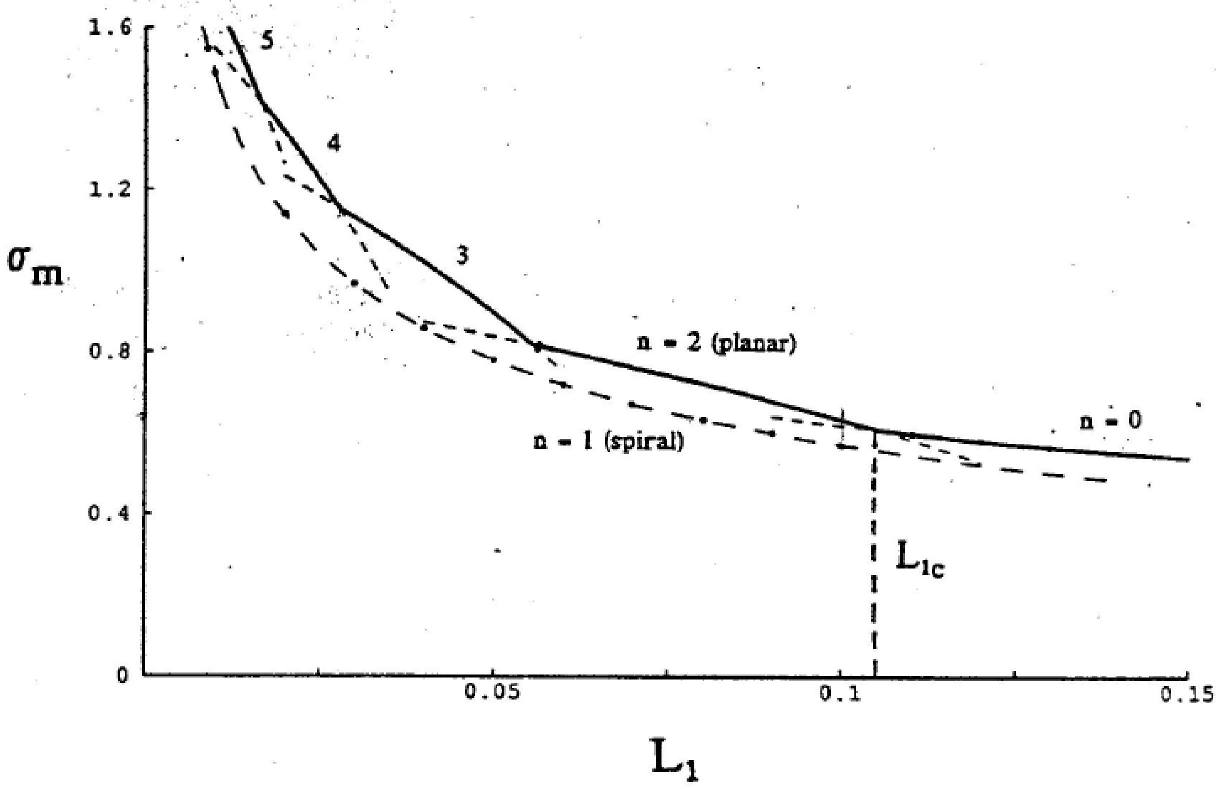


FIG. 4