The Fundamental Theorem of The Calculus

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22 May 2000

Abstract

The Fundamental Theorem of Calculus is appropriately named because is establishes a connection between the two branches of calculus: differential calculus and integral calculus.

1 The fundamental Theorem of Calculus, Part 1

Theorem 1. *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof. If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f - \int_a^x f \\ &= (\int_a^x f + \int_x^{x+h} f) - \int_a^x f \\ &= \int_x^{x+h} f \end{aligned}$$

and so, for $h \neq 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f \tag{1}$$

For now let us assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that three are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$ (see figure 1)

we have

$$mh \leq \int_x^{x+h} f \leq Mh$$

Figure 1: Absolute minimum and maximum values of f.

that is,

$$f(u)h \leq \int_x^{x+h} f \leq f(v)h$$

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2 The fundamental Theorem of Calculus, Part 2

Theorem 2. *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$

Proof. Let $g(x) = \int_a^x f$. We know form Part 1 that $g'(x) = g(x)$; then is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then We know that F and g differ by a constant:

$$F(x) = g(x) + C \tag{2}$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of Equation 2 (as $x \longrightarrow a^+$ and $x \longrightarrow b^-$), we see that it also holds when $x = a$ and $x = b$.

$$g(a) = \int_1^a f = 0$$

So using Equation 2 with $x = b$ and $x = a$, we have

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) - g(a) = g(b) = \int_a^b f \end{aligned}$$

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Example 1. *Evaluate the integral* $\int_{-2}^{1} x^3\,dx$.

Solution 1. *The function* $f(x) = x^3$ *is continuous on* $[-2, 1]$ *and we know that an antiderivative is* $F(x) = \frac{1}{4}x^4$, *so Part2 of the Fundamental Theorem gives*

$$\int_{-2}^{1} x^3 dx = F(1) - F(-2) = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = -\frac{15}{4}$$

References

[1] *J. Stewart. Calculus.* 2nd ed., early transcendentals, 1991.