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Pairwise balanced designs and sigma clique partitions

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ABSTRACT

In this paper, we are interested in minimizing the sum of block sizes in a pairwise balanced design, where there are some constraints on the size of one block or the size of the largest block. For every positive integers n, m, where $m \leq n$, let $\mathscr{S}(n, m)$ be the smallest integer s for which there exists a PBD on n points whose largest block has size m and the sum of its block sizes is equal to s. Also, let $\mathscr{S}'(n, m)$ be the smallest integer s for which there exists a PBD on n points which has a block of size m and the sum of it block sizes is equal to s. We prove some lower bounds for $\mathscr{S}(n, m)$ and $\mathscr{S}'(n, m)$. Moreover, we apply these bounds to determine the asymptotic behaviour of the sigma clique partition number of the graph $K_n - K_m$, the Cocktail party graphs and complement of paths and cycles.

1. Introduction

An $(n, k, \lambda)-$ *design* (or (n, k, λ)-BIBD) is a pair $(P, \mathscr{B})$ where P is a finite set of n *points* and $\mathscr{B}$ is a collection of $k-$subsets of P, called *blocks*, such that every two distinct points in P is contained in exactly λ blocks. In case $|P| = |\mathscr{B}|$, it is called a *symmetric design*. For positive integer q, a $(q^2+q+1, q+1, 1)$-BIBD and a $(q^2, q, 1)$-BIBD are called a *projective plane* and an *affine plane* of order q, respectively. A design is called *resolvable*, if there exists a partition of the set of blocks $\mathscr{B}$ into *parallel classes*, each of which is a partition of P.

A *pairwise balanced design* (PBD) is a pair $(P, \mathscr{B})$, where P is a finite set of n points and $\mathscr{B}$ is a family of subsets of P, called *blocks*, such that every two distinct points in P, appear in exactly one block. A *nontrivial* PBD is a PBD where $P \notin \mathscr{B}$. A PBD $(P, \mathscr{B})$ on n points with one block of size $n-1$ and the others of size two is called *near-pencil*.

The problem of determining the minimum number of blocks in a pairwise balanced design when the size of its largest block is specified or the size of a particular block is specified, has been the subject of many researches in recent decades. The most important and well-known result about this problem is due to de Bruijn and Erdős [3] which states that every nontrivial PBD on n points has at least n blocks and the only nontrivial PBDs on n points with exactly n blocks are near-pencil and projective plane. For every positive integers n, m, where $m \leq n$, let $\mathscr{G}(n, m)$ be the minimum number of blocks in a PBD on n points whose largest block has size m. Also, let $\mathscr{G}'(n, m)$ be the minimum number of blocks in a PBD on n points which has a block of size m. A classical result known as Stanton–Kalbfleisch Bound [14] states that $\mathscr{G}'(n, m) \geq 1 + (m^2(n-m))/(n-1)$ and equality holds if and only if there exists a resolvable $(n-m, (n-1)/m, 1)$- BIBD. Also, a corollary of Stanton–Kalbfleisch is that $\mathscr{G}(n, m) \geq \max\{n(n-1)/m(m-1), 1 + (m^2(n-m))/(n-1)\}$. For a survey on these and more bounds, see [12,13].

In this paper, we are interested in minimizing the sum of block sizes in a PBD, where there are some constraints on the size of one block or the size of the largest block. For every positive integers n, m, where $m \leq n$, let $\mathscr{S}(n, m)$ be the smallest integer s for which there exists a PBD on n points whose largest block has size m and the sum of its block sizes is equal to s. Also, let $\mathscr{S}'(n, m)$ be the smallest integer s for which there exists a PBD on n points which has a block of size m and the sum

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of its block sizes is equal to s. In Section 2, we prove some lower bounds for $\mathscr{S}(n, m)$ and $\mathscr{S}'(n, m)$. In particular, we show that $\mathscr{S}(n, m) \geq 3n - 3$, for every m, $2 \leq m \leq n - 1$. Also, we prove that, for every $2 \leq m \leq n$,

$$\mathscr{S}'(n, m) \geq \max\left\{(n+1)m - \frac{m^2(m-1)}{n-1}, m + \frac{(n-m)(n-5m-1)}{2}\right\},$$

where equality holds for $m \geq n/2$. Furthermore, we prove that if $n \geq 10$ and $2 \leq m \leq n - \frac{1}{2}(\sqrt{n} + 1)$, then $\mathscr{S}(n, m) \geq n(\lfloor\sqrt{n}\rfloor + 1) - 1$.

The connection of pairwise balanced designs and clique partition of graphs is already known in the literature. Given a simple graph G, by a *clique* in G we mean a subset of mutually adjacent vertices. A *clique partition* $\mathcal{C}$ of G is a family of cliques in G such that the endpoints of every edge of G lie in exactly one member of $\mathcal{C}$. The minimum size of a clique partition of G is called the *clique partition number* of G and is denoted by $\mathrm{cp}(G)$.

For every graph G with n vertices, the union of a clique partition of G and a clique partition of its complement, $\overline{G}$, form a PBD on n points. This connection has been deployed to estimate $\mathrm{cp}(G)$, when G is some special graph such as $K_n - K_m$ [15,4,8,11], the Cocktail party graphs and complement of paths and cycles [16–18].

Our motivation for study of the above mentioned problem is a weighted version of clique partition number. The *sigma clique partition number* of a graph G, denoted by $\mathrm{scp}(G)$, is defined as the smallest integer s for which there exists a clique partition of G where the sum of the sizes of its cliques is equal to s. It is known that for every graph G on n vertices, $\mathrm{scp}(G) \leq \lfloor n^2/2\rfloor$, in which equality holds if and only if G is the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ [2,7,6].

Given a clique partition $\mathcal{C}$ of a graph G, for every vertex $x \in V(G)$, the *valency* of x (with respect to $\mathcal{C}$), denoted by $v_{\mathcal{C}}(x)$, is defined to be the number of cliques in $\mathcal{C}$ containing x. In fact,

$$\mathrm{scp}(G) = \min_{\mathcal{C}} \sum_{C\in\mathcal{C}} |C| = \min_{\mathcal{C}} \sum_{x\in V(G)} v_{\mathcal{C}}(x),$$

where the minimum is taken over all possible clique partitions of G.

In Section 3, we apply the results of Section 2 to determine the asymptotic behaviour of the sigma clique partition number of the graph $K_n - K_m$, where m is a function of n. In fact, we prove that if $m \leq \sqrt{n}/2$, then $\mathrm{scp}(K_n - K_m) \sim (2m-1)n$, if $\sqrt{n}/2 \leq m \leq \sqrt{n}$, then $\mathrm{scp}(K_n - K_m) \sim n\sqrt{n}$ and if $m \geq \sqrt{n}$ and $m = o(n)$, then $\mathrm{scp}(K_n - K_m) \sim mn$. Also, if G is the Cocktail party graph, complement of path or cycle on n vertices, then we prove that $\mathrm{scp}(G) \sim n\sqrt{n}$.

2. Pairwise balanced designs

A celebrated result of de Bruijn and Erdős states that for every nontrivial PBD $(P, \mathscr{B})$, we have $|\mathscr{B}| \geq |P|$ and equality holds if and only if $(P, \mathscr{B})$ is near-pencil or projective plane [3]. In this section, we are going to answer the question that what is the minimum sum of block sizes in a PBD.

The following theorem can be viewed as a de Bruijn–Erdős-type bound, which shows that $\mathscr{S}(n, m) \geq 3n - 3$, for every m, $2 \leq m \leq n - 1$.

Theorem 2.1. *Let $(P, \mathscr{B})$ be a nontrivial PBD with n points, then we have*

$$\sum_{B\in\mathscr{B}} |B| \geq 3n - 3, \tag{1}$$

and equality holds if and only if $(P, \mathscr{B})$ is near-pencil.

Proof. We use induction on the number of points. Let $(P, \mathscr{B})$ be a nontrivial PBD with n points. Inequality (1) clearly holds when $n = 3$. So assume that $n \geq 4$ and for every $x \in P$, let r_x be the number of blocks containing x. First note that for every block $B \in \mathscr{B}$ and every $x \in P\backslash B$, we have $r_x \geq |B|$.

If there is a block $B_0 \in \mathscr{B}$ of size $n - 1$ and x_0 is the unique point in $P\backslash B_0$, then for every $x \in B_0$, x and x_0 appear within a block of size two. Therefore, $(P, \mathscr{B})$ is near-pencil and $\sum_{B\in\mathscr{B}} |B| = (n-1) + 2(n-1) = 3n - 3$.

Otherwise, all blocks are of size at most $n - 2$. First we prove that there exists some point $x \in P$ with $r_x \geq 3$. Since there is no block of size n, $r_x \geq 2$ for all $x \in P$. Now for some $y \in P$, assume that B_1, B_2 are the only two blocks containing y. Since $n \geq 4$, the size of at least one of these blocks, say B_1, is greater than two. Let $x \neq y$ be an element of B_2. Then, $r_x \geq |B_1| \geq 3$. Hence, there exists some point $x \in P$ which appears in at least three blocks.

Now, remove x from all blocks to obtain the nontrivial PBD $(P', \mathscr{B}')$, where $P' = P\backslash\{x\}$ and $\mathscr{B}' = \{B\backslash\{x\} : B \in \mathscr{B}\}$. Therefore,

$$\sum_{B\in\mathscr{B}} |B| = r_x + \sum_{B'\in\mathscr{B}'} |B'| \geq 3 + 3(n-2), \tag{2}$$

where the last inequality follows from the induction hypothesis.

Now, assume that for a PBD $(P, \mathscr{B})$ equality holds in (1). If $(P, \mathscr{B})$ is not a near-pencil, then equality holds in (2) as well and thus we have $2 \leq r_x \leq 3$, for every $x \in P$. On the other hand, $\sum_{B\in\mathscr{B}} |B| = \sum_{x\in P} r_x = 3n-3$. Therefore, there are exactly

3 points, say x, y, z, each of which appears in exactly two blocks and each of the other points appears in exactly three blocks. Also, let B_1, B_2 be the only two blocks containing y and assume that $x \in B_1$. Therefore, $2 = r_x \geq |B_2|$ and then $|B_1| = n - 1$, which is a contradiction. □

Since the union of every clique partition of G and $\overline{G}$ forms a clique partition for K_n which is equivalent to a PBD on n points, the following corollaries are straightforward.

Corollary 2.2. *Let $\mathcal{C}$ be a clique partition of K_n whose cliques are of size at most $n-1$. Then, $\sum_{C \in \mathcal{C}} |C| \geq 3n - 3$.*

Corollary 2.3. *For every graph G on n vertices except the empty and complete graph, we have*

$$\mathrm{scp}(G) + \mathrm{scp}(\overline{G}) \geq 3n - 3,$$

and equality holds if and only if G or $\overline{G}$ contains a clique of size $n-1$.

In the same vein, one can prove the following theorem which states a lower bound on the maximum number of appearances of the points in a PBD.

Theorem 2.4. *Let $(P, \mathcal{B})$ be a nontrivial PBD with n points, and for every $x \in P$, let r_x be the number of blocks containing x. Then, we have*

$$\max_{x \in P} r_x \geq \frac{1 + \sqrt{4n-3}}{2}, \tag{3}$$

and equality holds if and only if $(P, \mathcal{B})$ is a projective plane or near-pencil.

Proof. Let $(P, \mathcal{B})$ be a nontrivial PBD with n points and define $r = \max_{x \in P} r_x$. Fix a point $x \in P$ and let $\mathcal{B}_x \subset \mathcal{B}$ be the set of blocks containing x. The family of sets $\{B \backslash \{x\} \ : \ B \in \mathcal{B}_x\}$ is a partition of the set $P \backslash \{x\}$. Thus,

$$n - 1 = \sum_{B \in \mathcal{B}_x} (|B| - 1) \leq r_x (\max_{B \in \mathcal{B}_x} |B| - 1). \tag{4}$$

Therefore, there exists some block B_0 containing x, where $r_x(|B_0| - 1) \geq n - 1$. Now, let y be a point not in B_0. By a note within the proof of Theorem 2.1, we have $r_y \geq |B_0|$ and then

$$r(r-1) \geq r_x(r_y - 1) \geq r_x(|B_0| - 1) \geq n - 1. \tag{5}$$

This yields the inequality.

Now, assume that equality holds in (3). Then, we have equalities in (4) and (5). Thus, all valencies r_x are equal and all blocks have the same size, say k, which shows that $(P, \mathcal{B})$ is an $(n, k, 1)-$design. Also by (5), we have $r = k$, i.e. $(P, \mathcal{B})$ is a symmetric design. □

Although the given bound in (1) is sharp, it can be improved if the PBD avoids blocks of large sizes. The following theorem, as an improvement of Theorem 2.1, provides some lower bounds on the sum of block sizes, when there are some constraints on the size of a block.

Theorem 2.5. *If $(P, \mathcal{B})$ is a PBD with n points where τ is the maximum size of blocks in $\mathcal{B}$, then*

$$\sum_{B \in \mathcal{B}} |B| \geq \frac{n(n-1)}{\tau - 1}. \tag{6}$$

Also if there is a block of size k, then

$$\sum_{B \in \mathcal{B}} |B| \geq (n+1)k - \frac{k^2(k-1)}{n-1}, \tag{7}$$

and

$$\sum_{B \in \mathcal{B}} |B| \geq k - \frac{(n-k)(n-5k-1)}{2}. \tag{8}$$

Moreover, if $k \geq n/2$, then there exists a PBD on n points with a block of size k, for which equality holds in (8).

Proof. For every $x \in P$, let r_x be the number of blocks containing x. By inequality (4), we have

$$\sum_{B \in \mathcal{B}} |B| = \sum_{x \in P} r_x \geq \sum_{x \in P} \frac{n-1}{\tau - 1} = \frac{n(n-1)}{\tau - 1}.$$

In order to prove (7), let $B_0 \in \mathcal{B}$ and $|B_0| = k$. Define,

$$\tilde{\mathcal{B}} = \{B \backslash B_0 \ : \ B \in \mathcal{B}, B \cap B_0 \neq \emptyset\}.$$

We have

$$\sum_{B\in\tilde{\mathcal{B}}} |B| = k(n-k).$$

Now, consider the following set:

$$S = \{(x,y) : x \neq y, x, y \in B, B \in \tilde{\mathcal{B}}\}.$$

We have

$$|S| = \sum_{B\in\tilde{\mathcal{B}}} |B|(|B|-1) \geq \frac{1}{|\tilde{\mathcal{B}}|}\left(\sum_{B\in\tilde{\mathcal{B}}}|B|\right)^2 - \sum_{B\in\tilde{\mathcal{B}}}|B| = \frac{1}{|\tilde{\mathcal{B}}|}k^2(n-k)^2 - k(n-k). \tag{9}$$

On the other hand, $S \subseteq \{(x,y) \ : \ x, y \in P\backslash B_0\}$. Thus,

$$|S| \leq (n-k)(n-k-1). \tag{10}$$

Inequalities (9) and (10) yield

$$|\tilde{\mathcal{B}}| \geq \frac{k^2(n-k)}{n-1}.$$

Finally,

$$\sum_{B\in\mathcal{B}} |B| \geq |B_0| + \sum_{B\in\tilde{\mathcal{B}}}(|B|+1) \geq k + k(n-k) + \frac{k^2(n-k)}{n-1}.$$

Thus, we conclude

$$\sum_{B\in\mathcal{B}} |B| \geq (n+1)k - \frac{k^2(k-1)}{n-1}.$$

To prove Inequality (8), let $B_0 \in \mathcal{B}$ and $|B_0| = k$ and assume that $\mathcal{B}$ has u blocks of size 2 intersecting B_0. Define,

$$\hat{\mathcal{B}} = \{B\backslash B_0 \ : \ B \in \mathcal{B}, \ B \cap B_0 \neq \emptyset, \ |B| \geq 3\}.$$

Thus,

$$\binom{n-k}{2} \geq \sum_{B\in\hat{\mathcal{B}}} \binom{|B|}{2} \geq \sum_{B\in\hat{\mathcal{B}}}(|B|-1).$$

Also,

$$k(n-k) = u + \sum_{B\in\hat{\mathcal{B}}}|B|.$$

Hence,

$$\begin{aligned}\sum_{B\in\mathcal{B}} |B| &\geq |B_0| + 2u + \sum_{B\in\hat{\mathcal{B}}}(|B|+1) = k + 2k(n-k) - \sum_{B\in\hat{\mathcal{B}}}(|B|-1)\\ &\geq k + 2k(n-k) - \binom{n-k}{2}.\end{aligned}$$

Now, assume that $k \geq n/2$ and $B_0 = \{x_1, \ldots, x_k\}$. We provide a PBD with a block B_0 for which equality holds in (8). Consider a proper edge colouring of K_{n-k} by $n-k$ colours and let $C_1, \ldots, C_{n-k}$ be colour classes. Each C_i is a collection of subsets of size 2. For every i, $1 \leq i \leq n-k$, add x_i to each member of C_i. Now, we have exactly $(n-k)(n-k-1)/2$ blocks of size 3. By adding missing pairs as blocks of size 2, we get a PBD $(P, \mathcal{B})$ on n points, with blocks of size 2 and 3 and a block of size k. In fact, each block of size 3 contains two pairs from the set $\{(x,y) \ : \ x \in B_0, y \notin B_0\}$. Hence,

$$\begin{aligned}\sum_{B\in\mathcal{B}} |B| &= k + \frac{3(n-k)(n-k-1)}{2} + 2(k(n-k) - (n-k)(n-k-1))\\ &= k - \frac{(n-k)(n-5k-1)}{2}. \quad \square\end{aligned}$$

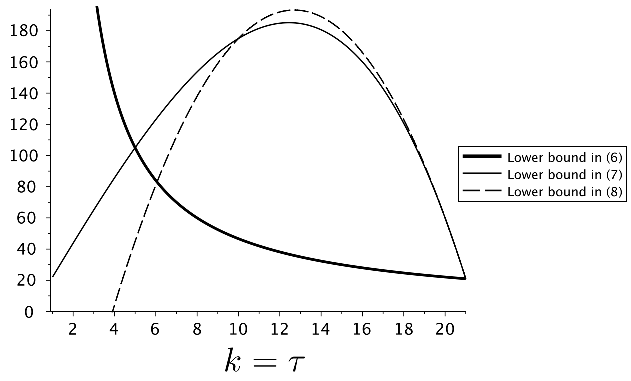

Fig. 1. Diagram of the lower bounds in (6)–(8) for $n = 21$.

Remark 2.6. Let $(P, \mathscr{B})$ be a PBD with n points where τ is the maximum size of blocks in $\mathscr{B}$. It is easy to check that among the lower bounds (6)–(8), if $1 \le \tau \le (\sqrt{4n-3}+1)/2$, then (6) is the best one, if $(\sqrt{4n-3}+1)/2 \le \tau \le (n-1)/2$, then (7) is the best one and if $(n-1)/2 \le \tau \le n-1$, then (8) is the best one. The diagram of the lower bounds in terms of τ are depicted in Fig. 1 for $n = 21$.

Now, we apply Theorem 2.5 to improve the bound in (1), whenever the PBD does not contain large blocks.

Theorem 2.7. *Let $n \ge 10$ and $(P, \mathscr{B})$ be a PBD on n points and assume that $\mathscr{B}$ contains no block of size larger than $n-\frac{1}{2}(\sqrt{n}+1)$. Then, we have*

$$\sum_{B\in\mathscr{B}} |B| \ge n(\lfloor\sqrt{n}\rfloor + 1) - 1.$$

Also, the bound is tight in the sense that equality occurs for infinitely many n.

Proof. Let τ be the maximum size of the blocks in $\mathscr{B}$. If $\tau \le \sqrt{n}$, then by (6),

$$\sum_{B\in\mathscr{B}} |B| \ge \frac{n(n-1)}{\tau-1} \ge \frac{n(n-1)}{\sqrt{n}-1} \ge n(\sqrt{n}+1).$$

Now, suppose that $\tau \ge \lfloor\sqrt{n}\rfloor + 1$. Then, $\mathscr{B}$ contains a block of size larger than or equal to $\lfloor\sqrt{n}\rfloor + 1$. First assume that $\mathscr{B}$ contains a block of size k, where $\lfloor\sqrt{n}\rfloor + 1 \le k \le \frac{n}{2}$. Then, by (7),

$$\sum_{B\in\mathscr{B}} |B| \ge (n+1)k - \frac{k^2(k-1)}{n-1}.$$

The right hand side of the above inequality as a function of k takes its minimum on the interval $[\lfloor\sqrt{n}\rfloor + 1, \frac{n}{2}]$ at $\lfloor\sqrt{n}\rfloor + 1$. Thus,

$$\begin{aligned}\sum_{B\in\mathscr{B}} |B| &\ge (n+1)(\lfloor\sqrt{n}\rfloor + 1) - \frac{(\lfloor\sqrt{n}\rfloor + 1)^2\lfloor\sqrt{n}\rfloor}{n-1} \\ &\ge n(\lfloor\sqrt{n}\rfloor + 1) + (\lfloor\sqrt{n}\rfloor + 1)\left(1 - \frac{(\sqrt{n}+1)\sqrt{n}}{n-1}\right) \\ &= n(\lfloor\sqrt{n}\rfloor + 1) - \frac{\lfloor\sqrt{n}\rfloor + 1}{\sqrt{n}-1} \\ &> n(\lfloor\sqrt{n}\rfloor + 1) - 2.\end{aligned}$$

The last inequality holds, since $n \ge 10$. Finally, assume that $\mathscr{B}$ contains a block of size k, where $\frac{n}{2} < k \le n - \frac{1}{2}(\sqrt{n}+1)$. Then, by (8)

$$\sum_{B\in\mathscr{B}} |B| \ge k - \frac{(n-k)(n-5k-1)}{2}.$$

Again, the right hand side of the above inequality as a function of k takes its minimum on the interval $[\frac{n}{2}, n - \frac{1}{2}(\sqrt{n}+1)]$ at $n - \frac{1}{2}(\sqrt{n}+1)$. Hence,

$$\begin{aligned}\sum_{B\in\mathcal{B}} |B| &\geq n - \frac{1}{2}(\sqrt{n}+1) - \frac{(\sqrt{n}+1)(-4n + \frac{5}{2}(\sqrt{n}+1) - 1)}{4} \\ &= n(\sqrt{n}+1) + \frac{3n-7}{8} - \frac{3}{2}\sqrt{n} \\ &> n(\sqrt{n}+1) - 2,\end{aligned}$$

where the last inequality is because $n \geq 10$. This completes the proof.

Finally, in order to prove tightness of the bound, let q be a prime power and $(P, \mathcal{B})$ be an affine plane of order q. Suppose that $\{B_1, \dots, B_q\}$ is a parallel class. Add a single new point to all the blocks $B_1, \dots, B_q$. The new PBD has $n = q^2 + 1$ points, q^2 blocks of size q and q blocks of size $q + 1$. Hence, the sum of its block sizes is

$$q^3 + q^2 + q = (q^2 + 1)(q + 1) - 1 = n(\lfloor\sqrt{n}\rfloor + 1) - 1. \quad \square$$

3. Sigma clique partition of complement of graphs

Given a graph G and its subgraph H, the complement of H in G denoted by $G - H$ is obtained from G by removing all edges (but no vertices) of H. If H is a graph on n vertices, then $K_n - H$ is called the complement of H and is denoted by $\overline{H}$.

In this section, applying the results of Section 2, we are going to determine the asymptotic behaviour of the sigma clique partition number of the graph $K_n - K_m$, when m is a function of n, as well as the Cocktail party graph, the complement of path and cycle on n vertices.

The clique partition number of the graph $K_n - K_m$, for $m \leq n$, has been studied by several authors. In order to notice the hardness of determining the exact value of $\mathrm{cp}(K_n - K_m)$, note that if we could show that $\mathrm{cp}(K_{111} - K_{11}) \geq 111$, then we could determine whether there exists a projective plane of order 10 [8]. Wallis in [15], proved that $\mathrm{cp}(K_n - G) \sim n$, if G has $o(\sqrt{n})$ vertices. Also, Erdős et al. in [4] showed that $\mathrm{cp}(K_n - K_m) \sim m^2$, if $\sqrt{n} < m < n$ and $m = o(n)$. Moreover, if $m = cn$ and $1/2 \leq c \leq 1$, then Pullman et al. in [9] proved that $\mathrm{cp}(K_n - K_m) = 1/2(n - m)(3m - n - 1)$.

In the following theorem, we present upper and lower bounds for $\mathrm{scp}(K_n - K_m)$ and then we improve these bounds in order to determine asymptotic behaviour of $\mathrm{scp}(K_n - K_m)$.

Theorem 3.1. *For every m, n, $1 \leq m \leq n$, we have*

$$mn - \frac{m^2(m-1)}{n-1} \leq \mathrm{scp}(K_n - K_m) \leq (2m-1)(n-m) + 1. \tag{11}$$

Proof. Adding the clique K_m to every clique partition of $K_n - K_m$ forms a PBD on n points. Thus, the lower bound is obtained from Inequality (7). For the upper bound, let $V(K_n) = \{x_1, \dots, x_n\}$ and $V(K_m) = \{x_{n-m+1}, \dots, x_n\}$. Note that the clique $\{x_1, \dots, x_{n-m+1}\}$ along with $(m-1)(n-m)$ remaining edges form a clique partition of $K_n - K_m$. Hence, $\mathrm{scp}(K_n - K_m) \leq (n - m + 1) + 2(m - 1)(n - m)$. $\square$

In the following theorem, for $m \leq \frac{\sqrt{n}}{2}$, we improve the lower bound in (11).

Theorem 3.2. *If $m \leq \frac{\sqrt{n}}{2}$, then*

$$(2m-1)n - O(m^2) \leq \mathrm{scp}(K_n - K_m) \leq (2m-1)n - \Omega(m^2).$$

Proof. The upper bound holds by (11). For the lower bound, consider an arbitrary clique partition of $K_n - K_m$, say $\mathcal{C}$, and add the clique K_m to obtain a PBD $(P, \mathcal{B})$ with n points. Let τ be the size of maximum block in $\mathcal{B}$. It is clear that $m \leq \tau \leq n - m + 1$. We give the lower bound in the following cases. First note that since $m \leq \sqrt{n}/2$, we have $(2m-1)^2 \leq n - 1$.

If $\tau \leq \frac{n-1}{2m-1}$, then by (6), we have

$$\sum_{C\in\mathcal{C}} |C| \geq (2m-1)n - m.$$

If $\frac{n-1}{2m-1} \leq \tau \leq n/2$, then $2m - 1 \leq \tau \leq n/2$, and by (7),

$$\sum_{C\in\mathcal{C}} |C| \geq (n+1)\tau - \frac{\tau^2(\tau-1)}{n-1} - m.$$

The right hand side of this inequality is increasing as a function of τ within the interval $[2m-1, n/2]$. Hence,

$$\sum_{C\in\mathcal{C}} |C| \geq (n+1)(2m-1) - \frac{(2m-1)^2(2m-2)}{n-1} - m \geq (2m-1)n - m.$$

Finally, if $n/2 \leq \tau \leq n-m+1$, then, by (8),

$$\sum_{C\in\mathcal{C}} |C| \geq \tau - \frac{(n-\tau)(n-5\tau-1)}{2} - m.$$

Consider the right hand side of this inequality as a function of τ within the interval $[n/2, n-m+1]$. It attains its minimum at $\tau = n-m+1$. Hence,

$$\sum_{C\in\mathcal{C}} |C| \geq n-2m+1 - \frac{(m-1)(5m-4n-6)}{2} = (2m-1)n - O(m^2). \quad \square$$

The following lemma is a direct application of Theorem 2.7 that gives a lower bound for $\mathrm{scp}(K_n - H)$ in terms of $\mathrm{scp}(H)$. Here, $\omega(G)$ stands for the clique number of graph G.

Lemma 3.3. *Let H be a graph on m vertices. If $\omega(H) \leq n - \frac{1}{2}(\sqrt{n}+1)$ and $\omega(\overline{H}) \leq m - \frac{1}{2}(\sqrt{n}+1)$, then*

$$\mathrm{scp}(K_n - H) + \mathrm{scp}(H) \geq n(\lfloor\sqrt{n}\rfloor + 1) - 1.$$

Proof. Assume that $\mathcal{C}$ is an arbitrary clique partition for $K_n - H$ and τ is the size of largest clique in $\mathcal{C}$. Then, we have $\tau \leq n - m + \omega(\overline{H}) \leq n - m + m - \frac{1}{2}(\sqrt{n}+1) = n - \frac{1}{2}(\sqrt{n}+1)$. Also, by assumption, H has no clique of size larger than $n - \frac{1}{2}(\sqrt{n}+1)$. Moreover, every clique partition of H along with every clique partition for $K_n - H$ form a PBD. Hence, by Theorem 2.7, $\mathrm{scp}(K_n - H) + \mathrm{scp}(H) \geq n(\lfloor\sqrt{n}\rfloor + 1) - 1$. $\square$

We need the following lemma in order to improve the upper bound in (11) whenever $\sqrt{n} \leq m \leq n$. The idea is similar to [15] that uses a projective plane of appropriate size to give a clique partition for the graph $K_n - K_m$.

Lemma 3.4. *Let H be a graph on m vertices. If there exists a $(v, k, 1)-$ design, such that $k \geq m$ and $v - k \geq n - m$, then $\mathrm{scp}(K_n - H) \leq n(v-1)/(k-1) + \mathrm{scp}(\overline{H}) - m$.*

Proof. Let $(P, \mathcal{B})$ be a $(v, k, 1)-$design. Select a block $B_1 \in \mathcal{B}$ and delete $k - m$ points from it. Also, delete $v - k - (n-m)$ points not in B_1. Now, consider the remaining points as vertices of $K_n - H$ and each block except B_1 as a clique in $K_n - H$. Thus, $\mathrm{scp}(K_n - H) \leq r(n-m) + (r-1)m + \mathrm{scp}(\overline{H}) = nr - m + \mathrm{scp}(\overline{H})$, where $r = (v-1)/(k-1)$ is the number of blocks containing a single point. $\square$

We are going to apply Lemma 3.4 to projective planes and provide a clique covering for $K_n - H$. Since the existence of projective planes of order q is only known for prime powers, we need the following well-known theorem to approximate an integer by a prime.

Theorem A ([1]). *There exists a constant x_0 such that for every integer $x > x_0$, the interval â $[x, x + x^{.525}]$ contains prime numbers.*

The following two theorems determine asymptotic behaviour of $\mathrm{scp}(K_n - K_m)$, when $\sqrt{n}/2 \leq m$ and $m = o(n)$.

Theorem 3.5. *Let H be a graph on m vertices. If $\frac{\sqrt{n}}{2} \leq m \leq \sqrt{n}$, then $\mathrm{scp}(K_n - H) \leq (1 + o(1))\, n\sqrt{n}$. Moreover, $\mathrm{scp}(K_n - K_m) = (1 + o(1))\, n\sqrt{n}$.*

Proof. Let q be the smallest prime power greater than or equal to $\sqrt{n}$. By Theorem A, we have $\sqrt{n} \leq q \leq \sqrt{n} + \sqrt{n}^{.525}$. Thus, $q \geq \sqrt{n} > m - 1$ and $q^2 \geq n \geq n - m$. Since there exists a projective plane of order q, by Lemma 3.4, we have

$$\mathrm{scp}(K_n - H) \leq n(q+1) - m + \mathrm{scp}(\overline{H}) \leq n(q+1) - m + \frac{m^2}{2},$$

where the last inequality is due to the fact that for every graph G on n vertices, $\mathrm{scp}(G) \leq \frac{n^2}{2}$ [2,6]. Hence,

$$\mathrm{scp}(K_n - H) \leq n^{1.5} + n^{1.2625} + 1.5\, n = (1 + o(1))\, n\sqrt{n}.$$

Also, by Lemma 3.3, $\mathrm{scp}(K_n - K_m) \geq (1 + o(1))\, n\sqrt{n}$. $\square$

In the following theorem, for $\sqrt{n} \leq m \leq n$, we improve the upper bound in (11).

Theorem 3.6. *If $\sqrt{n} \le m \le n$, then $\mathrm{scp}(K_n - K_m) \le (1 + o(1))\, nm$. Also, if in addition $m = o(n)$, then $\mathrm{scp}(K_n - K_m) = (1 + o(1))\, nm$.*

Proof. Let $\sqrt{n} \le m \le n$, and also let q be the smallest prime power which is greater than or equal to m. By Lemma A, $m \le q \le m + m^{.525}$. Thus, $q = (1 + o(1))\, m$. Since there exists a projective plane of order q, by Lemma 3.4, we have

$$\mathrm{scp}(K_n - K_m) \le n(q + 1) - m = (1 + o(1))\, nm.$$

On the other hand, when $m = o(n)$, Inequality (11) yields $\mathrm{scp}(K_n - K_m) \ge (1 + o(1))\, nm$, which completes the proof. □

Theorems 3.2, 3.5 and 3.6 make clear asymptotic behaviour of $K_n - K_m$ in case $m = o(n)$.

Corollary 3.7. *Let m be a function of n. Then*

(i) *If $m \le \frac{\sqrt{n}}{2}$, then $\mathrm{scp}(K_n - K_m) \sim (2m - 1)n$.*
(ii) *If $\frac{\sqrt{n}}{2} \le m \le \sqrt{n}$, then $\mathrm{scp}(K_n - K_m) \sim n\sqrt{n}$.*
(iii) *If $m \ge \sqrt{n}$ and $m = o(n)$, then $\mathrm{scp}(K_n - K_m) \sim mn$.*

In what follows, we consider the case $m = cn$, where c is a constant. First note that if $1/2 \le c \le 1$, then by Theorem 2.5, since $m \ge n/2$, there exists a PBD on n points with a block of size m, for which equality holds in (8). Hence, we have $\mathrm{scp}(K_n - K_m) = \frac{(1-c)}{2}\left((5c - 1)n^2 + n\right)$. In order to deal with the case $c < 1/2$, we need the following well-known existence theorem of resolvable designs.

Theorem B ([10]). *Given any integer $k \ge 2$, there exists an integer $v_0(k)$ such that for every $v \ge v_0(k)$, a $(v, k, 1)-$ resolvable design exists if and only if $v \overset{k}{\equiv} 0$ and $v - 1 \overset{k-1}{\equiv} 0$.*

Theorem 3.8. *Let $0 < c < 1/2$ be a constant and m, n be some integers satisfying $m = cn$. Then*

$$c(1 - c^2)n^2 + \Omega(n) \le \mathrm{scp}(K_n - K_m) \le \frac{(1 - c)(\lfloor 1/c \rfloor - c)}{\lfloor 1/c \rfloor(\lfloor 1/c \rfloor - 1)} n^2 + O(n). \tag{12}$$

In particular, if $1/c$ is integer, then $\mathrm{scp}(K_n - K_m) \sim c(1 - c^2)n^2$.

Proof. The lower bound in (12) is obtained from the lower bound in (11). For the upper bound, let $k = \lfloor 1/c \rfloor$ and define v as the smallest number greater than or equal to $n - m$ which satisfies the conditions of Theorem B. Without loss of generality we can assume that n is sufficiently large, i.e. $n \ge v_0(k)$. Thus, we have $v \le n - m + k^2$ and by Theorem B, there exists a $(v, k, 1)-$resolvable design. Remove $v - n + m$ points from such a design to obtain a PBD $(P, \mathcal{B})$ on $n - m$ points whose blocks are partitioned into $t = (v - 1)/(k - 1)$ parallel classes. First, we show that $m \le t$. Note that

$$m - t = cn - \frac{v - 1}{k - 1} \le cn - \frac{(1 - c)n - 1}{k - 1} = \frac{(ck - 1)n + 1}{k - 1}.$$

If $k = 2$, then $ck < 1$ and $m - t < 1$. Also, if $k > 2$, then $ck \le 1$ and thus $m - t \le 1/(k - 1) < 1$. Therefore, $m \le t$.

Now, let $v_1, \ldots, v_m$ be m new points and for every i, $1 \le i \le m$, add point v_i to all blocks of ith parallel class. These blocks form a clique partition $\mathcal{C}$ for $K_n - K_m$, where

$$\sum_{C \in \mathcal{C}} |C| \le \sum_{B \in \mathcal{B}} |B| + \frac{v}{k} m = (n - m)\frac{v - 1}{k - 1} + \frac{mv}{k}.$$

Hence,

$$\sum_{C \in \mathcal{C}} |C| \le \left(\frac{(1 - c)^2}{k - 1} + \frac{c(1 - c)}{k}\right) n^2 + O(n)$$
$$= \frac{(1 - c)(k - c)}{k(k - 1)} n^2 + O(n). \quad \square$$

We close the paper by proving that if G is the Cocktail party graph, complement of path or cycle on n vertices, then $\mathrm{scp}(G) \sim n\sqrt{n}$. Given an even positive integer n, the Cocktail party graph T_n is obtained from the complete graph K_n by removing a perfect matching. If n is an odd positive integer, then T_n is obtained from T_{n+1} by removing a single vertex. In [18,5] it is proved that if G is the Cocktail party graph or complement of a path or a cycle on n vertices, then $n \le \mathrm{cp}(G) \le (1 + o(1))\, n \log\log n$ and it is conjectured that for such a graph, $\mathrm{cp}(G) \sim n$.

Theorem 3.9. *Let P_n be the path on n vertices. Then, $\mathrm{scp}(\overline{P_n}) \sim n^{3/2}$.*

Proof. By Lemma 3.3, we have $\operatorname{scp}(\overline{P_n}) \geq n^{3/2} - 2n - 3$. Now, by induction on n, we prove that there exists a constant c, such that $\operatorname{scp}(\overline{P_n}) \leq n^{3/2} + c\, n^{13/10}$. The idea is similar to [18].

Let $d = \lfloor \sqrt{n} \rfloor$, $e = \lceil \frac{n}{d} \rceil$ and q be the smallest prime greater than $\sqrt{n}$. By Lemma A, $q \leq \sqrt{n} + n^{3/10}$. In an affine plane of order q, choose a parallel class, say C_1, and delete $q - d$ blocks in C_1. Then, remove $q - e$ blocks in a second parallel class, say C_2. The collection of remaining blocks is a PBD on de points.

Assume that a_{ij} is the intersection point of block i of C_1 and block j of C_2 in the remaining PBD. Thus, $C_1 = \{\{a_{i1}, a_{i2}, \ldots, a_{ie}\} : 1 \leq i \leq d\}$ and $C_2 = \{\{a_{1j}, a_{2j}, \ldots, a_{dj}\} : 1 \leq j \leq e\}$. Now, replace each block in C_2 by members of a clique partition of a copy of $\overline{P_d}$ on the same vertices. Also, replace each of the blocks $\{a_{11}, a_{12}, \ldots, a_{1e}\}$ and $\{a_{d1}, a_{d2}, \ldots, a_{de}\}$ in C_1 by members of a clique partition of a copy of $\overline{P_e}$ on the same vertices. In fact, we have replaced $e + 2$ blocks by some clique partitions of complement of paths and $q(q + 1) - (e + 2)$ blocks are left unchanged. It can be seen that the resulting collection, is a partition of all edges of $\overline{P_{de}}$ except $(e - 1)$ edges namely $a_{11}a_{12}, a_{d2}a_{d3}, a_{13}a_{14}, a_{d4}a_{d5}, \ldots$. Adding these $e - 1$ edges to this collection comprise a clique partition for $\overline{P_{de}}$. Hence,

$$\operatorname{scp}(\overline{P_n}) \leq \operatorname{scp}(\overline{P_{de}}) \leq qde - 2e + e \operatorname{scp}(\overline{P_d}) + 2 \operatorname{scp}(\overline{P_e}) + 2(e - 1).$$

Since $e \leq d + 3$, $\operatorname{scp}(\overline{P_e}) \leq \operatorname{scp}(\overline{P_d}) + 6d$. Thus,

$$\operatorname{scp}(\overline{P_n}) \leq qd(d + 3) + (d + 5) \operatorname{scp}(\overline{P_d}) + 12d.$$

Therefore, by the induction hypothesis, we have

$$\begin{aligned} \operatorname{scp}(\overline{P_n}) &\leq (\sqrt{n} + n^{3/10})\sqrt{n}(\sqrt{n} + 3) + (\sqrt{n} + 5)(n^{3/4} + c\, n^{13/20}) + 12\sqrt{n} \\ &\leq n^{3/2} + (1 + o(1))\, n^{13/10} \\ &\leq n^{3/2} + c\, n^{13/10}. \quad \square \end{aligned}$$

Asymptotic behaviour of $\operatorname{scp}(T_n)$ and $\operatorname{scp}(\overline{C_n})$ can be easily determined using $\operatorname{scp}(\overline{P_n})$, as follows.

Corollary 3.10. *Let T_n and C_n be the Cocktail party graph and cycle on n vertices, respectively. Then,* $\operatorname{scp}(\overline{C_n}) \sim n^{3/2}$ *and* $\operatorname{scp}(T_n) \sim n^{3/2}$.

Proof. By Lemma 3.3, $\operatorname{scp}(\overline{C_n}) \geq n^{3/2} - 2n - 1$ and $\operatorname{scp}(T_n) \geq n^{3/2} - n - 1$.

Note that $\overline{P_n}$ is obtained from $\overline{C_{n+1}}$ by removing an arbitrary vertex v. Adding $n - 2$ edges incident with v to any clique partition of $\overline{P_n}$ forms a clique partition for $\overline{C_{n+1}}$. Therefore, $\operatorname{scp}(\overline{C_{n+1}}) \leq \operatorname{scp}(\overline{P_n}) + 2(n - 1)$. Also, adding at most $n/2$ edges to any clique partition for $\overline{P_n}$ forms a clique partition for T_n. Thus, $\operatorname{scp}(T_n) \leq \operatorname{scp}(\overline{P_n}) + 2\frac{n}{2}$. Hence, by Theorem 3.9, $\operatorname{scp}(\overline{C_n}), \operatorname{scp}(T_n) \leq (1 + o(1))\, n^{3/2}$. $\square$

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