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A characterization of some graphs with metric dimension two

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A set $W \subseteq V(G)$ is called a resolving set, if for each pair of distinct vertices $u, v \in V(G)$ there exists $t \in W$ such that $d(u,t) \neq d(v,t)$, where $d(x,y)$ is the distance between vertices x and y. The cardinality of a minimum resolving set for G is called the metric dimension of G and is denoted by $\dim_M(G)$. A k-tree is a chordal graph all of whose maximal cliques are the same size $k+1$ and all of whose minimal clique separators are also all the same size k. A k-path is a k-tree with maximum degree $2k$, where for each integer j, $k \leq j < 2k$, there exists a unique pair of vertices, u and v, such that $\deg(u) = \deg(v) = j$. In this paper, we prove that if G is a k-path, then $\dim_M(G) = k$. Moreover, we provide a characterization of all 2-trees with metric dimension two.

Keywords: Resolving set; metric dimension; metric basis; k-tree.

Mathematics Subject Classification 2010: 05C75, 05C12

1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. The notions δ, Δ and $N_G(v)$ stand for minimum degree, maximum degree and the set of neighbors of vertex v in G, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the k-vector $r(v \mid W) := (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$ is called the *metric representation* of v with respect to W, where $d(x,y)$ is the distance between two vertices x and y. The set W is called a *resolving set* for G if distinct vertices of G have distinct representations with respect to W. We say a set

$S \subseteq V(G)$ *resolves* a set $T \subseteq V(G)$ if for each pair of distinct vertices u and v in T there is a vertex $s \in S$ such that $d(u,s) \neq d(v,s)$. A minimum resolving set is called a *basis* and the *metric dimension* of G, $\dim_M(G)$, is the cardinality of a basis for G. A graph with metric dimension k is called k-*dimensional*.

The concept of the resolving set has various applications in diverse areas including coin weighing problems [10], network discovery and verification [1], robot navigation [8], mastermind game [3], problems of pattern recognition and image processing [9], and combinatorial search and optimization [10].

These concepts were introduced by Slater in [11]. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [6] discovered these concepts. In [8], it is proved that determining the metric dimension of a graph in general is an NP-complete problem, but the metric dimension of trees can be obtained by a polynomial time algorithm.

It is obvious that for every graph G of order n, $1 \leq \dim_M(G) \leq n-1$. Chartrand *et al.* [4] proved that for $n \geq 2$, $\dim_M(G) = n-1$ if and only if G is the complete graph K_n. They also provided a characterization of graphs of order n and metric dimension $n-2$ [4]. Graphs with metric dimension $n-3$ are characterized in [7]. Khuller *et al.* [8] and Chartrand *et al.* [4] proved that $\dim_M(G) = 1$ if and only if G is a path. Moreover, in [12] some properties of two-dimensional graphs are obtained.

Theorem 1.1 ([12]). *Let G be a two-dimensional graph. If $\{a,b\}$ is a basis for G, then*

(1) *There is a unique shortest path P between a and b,*
(2) *The degrees of a and b are at most three,*
(3) *The degree of each internal vertex on P is at most five.*

A *chordal graph* is a graph with no induced cycle of length greater than three. A k-*tree* is a chordal graph that all of whose maximal cliques are the same size $k+1$ and all of whose minimal clique separators are also all the same size k. In other words, a k-tree may be formed by starting with a set of $k+1$ pairwise adjacent vertices and then repeatedly adding vertices in such a way that each added vertex has exactly k neighbors that form a k-clique.

By the above definition, it is clear that if G is a k-tree, then $\delta(G) = k$. 1-trees are the same as trees; 2-trees are maximal series-parallel graphs [5] and include also the maximal outerplanar graphs. These graphs can be used to model series and parallel electric circuits. Planar 3-trees are also known as Apollonian networks [2].

A k-*path* is a k-tree with maximum degree $2k$, where for each integer j, $k \leq j < 2k$, there exists a unique pair of vertices, u and v, such that $\deg(u) = \deg(v) = j$. On the other hand, regards to the recursive construction of k-trees, a k-path G can be considered as a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{v_iv_j : |i-j| \leq k\}$. For instance, two different representations of a 2-path G with seven vertices $v_1, \ldots, v_7$ are shown in Fig. 1.

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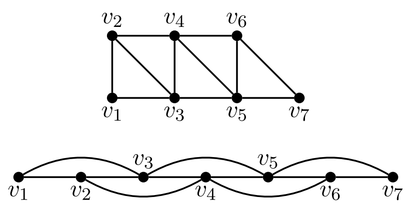

Fig. 1. Two different representations of a 2-path.

In this paper, we show that the metric dimension of each k-path (as a generalization of a path) is k. Whereas, there are some examples of 2-trees with metric dimension two that are not 2-path. This fact motivates us to study the structure of two-dimensional 2-trees. As a main result, we characterize the class of all 2-trees with metric dimension two.

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2. Main Results

In this section, we first prove that the metric dimension of each k-path is k. Then, we introduce a class of graphs which shows that the inverse of this fact is not true in general. Later on, we concern on the case $k = 2$ and toward to investigating all 2-trees with metric dimension two, we construct a family $\mathcal{F}$ of 2-trees with metric dimension two. Finally, as the main result, we prove that the metric dimension of a 2-tree G is two if and only if G belongs to $\mathcal{F}$.

Theorem 2.1. *If G is a k-path, then* $\dim_M(G) = k$.

Proof. Let G be a k-path with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{v_iv_j : |i-j| \leq k\}$. Therefore, the distance between two vertices v_r and v_s in G is given by $d(v_r, v_s) = \lceil \frac{|r-s|}{k} \rceil$.

At first, let $W = \{v_1, v_2, \ldots, v_k\}$ and v_i, v_j be two distinct vertices of G with $k < i < j$. By the division algorithm, there exist integers r and s such that $i = rk+s$, $1 \leq s \leq k$. Thus, we have

$$d(v_i, v_s) = \left\lceil \frac{|i-s|}{k} \right\rceil = \left\lceil \frac{rk}{k} \right\rceil = r,$$

and

$$d(v_j, v_s) = \left\lceil \frac{|j-s|}{k} \right\rceil = \left\lceil \frac{rk + (j-i)}{k} \right\rceil = r + \left\lceil \frac{j-i}{k} \right\rceil \geq r+1.$$

This means W is a resolving set for G. Hence, $\dim_M(G) \leq |W| = k$.

Now, we show that $\dim_M(G) \geq k$. Let W be a basis of the k-path G, and let $X = \{v_1, v_2, \ldots, v_{k+1}\}$. Assume that $|W \cap X| = s$ and $X \backslash W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{k+1-s}}\}$, where $1 \leq i_1 < i_2 < \cdots < i_{k+1-s} \leq k+1$. For convince, let $X' = \{x_1, x_2, \ldots, x_{k+1-s}\}$, where $x_r = v_{i_r}$, for each r, $1 \leq r \leq k+1-s$. Since each vertex v_i of the k-path G is adjacent to the next k consecutive vertices $\{v_{i+1}, \ldots, v_{i+k}\}$, the induced subgraph on X is a $(k+1)$-clique. Each vertex in $W \cap X$ is adjacent to each vertex in X'. Thus, each pair of vertices in X' should be resolved by some element of $W \backslash X$. Assume that $W' = \{w_1, w_2, \ldots, w_t\}$ is a minimum subset of $W \backslash X$ which resolves vertices in X'. Thus, for each $w_j \in W'$ there exists $\{x_r, x_s\} \subseteq X'$ such that $d(w_j, x_r) \neq d(w_j, x_s)$. For each j, $1 \leq j \leq t$, let

$$r_j = \min\{r : d(w_j, x_r) \neq d(w_j, x_{r+1})\},$$

and, let

$$A_j = \{x_1, x_2, \ldots, x_{r_j}\},\ B_j = \{x_{r_j+1}, x_{r_j+2}, \ldots, x_{k+1-s}\}.$$

Note that $A_j \cup B_j = X'$, $A_j \cap B_j = \emptyset$, $x_1 \in A_j$ and $x_{k+1-s} \in B_j$. Also, the structure of G implies that

$$d(w_j, x_1) = d(w_j, x_2) = \cdots = d(w_j, x_{r_j}),$$

and

$$d(w_j, x_{r_j+1}) = d(w_j, x_{r_j+2}) = \cdots = d(w_j, x_{k+1-s}).$$

Since W' has the minimum size, for each $1 \leq j < j' \leq t$, we have $A_j \neq A_{j'}$ (otherwise, w_j and $w_{j'}$ resolve the same pair of vertices in X') and hence, $|A_j| \neq |A_{j'}|$. Moreover, for each r, $1 \leq r \leq k-s$, there exists $w_j \in W'$ such that $d(w_j, x_r) \neq d(w_j, x_{r+1})$ which implies $|A_j| = r$. Therefore,

$$t = |\{|A_1|, |A_2|, \ldots, |A_t|\}| = |\{1, 2, \ldots, k-s\}| = k-s.$$

Hence,

$$|W| = |W \backslash X| + |W \cap X| \geq |W'| + s = (k-s) + s = k,$$

which completes the proof. □

Definition 2.2. Let G and H be two 2-trees. We say that H is a *branch* in G on $\{u, v\}$, for convenience say a (u, v)-branch, if $V(H) \cap V(G) = \{u, v\}$, where uv is an edge of G belonging to only one of the triangles in H. The *length* of a branch in a 2-tree is the number of it's triangles, which is equal to the number of vertices of the branch minus 2. A cane is a 2-path with a branch of length one on a specific edge as shown in Fig. 2.

In the following proposition, we provide some 2-trees with metric dimension two other than 2-paths.

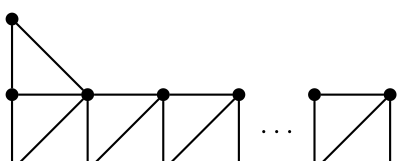

Fig. 2. A cane.

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Proposition 2.3. *If G is a 2-tree of metric dimension two with a basis whose elements are adjacent, then G is a 2-path or a cane.*

Proof. We prove the statement by induction on n, the order of G. If $n = 3$, then $G = K_3$ and the statement holds. Let G be a 2-tree of order $n > 3$ with a basis $B = \{a, b\}$, such that $d(a, b) = 1$. Since each 2-tree of order greater than three has two non-adjacent vertices of degree two, there exists a vertex $x \in V(G)\backslash B$ of degree two. Moreover, B is a basis for $G\backslash\{x\}$.

Now, by the induction hypothesis, $G\backslash\{x\}$ is a path or a cane and by Theorem 1.1(2), the degrees of a and b are at most three. Therefore, $B = \{a, b\}$ is one of the possible cases shown in Fig. 3. Note that dashed edges could be absent. It can be checked that in cases (b) and (c) the bold vertices get the same metric representation with respect to B. Thus, B is one of the cases (a) or (d), where the metric representations of vertices are denoted in Fig. 3.

Regards to the metric representation of vertices in G, x could be adjacent to the vertices by metric representation $(t, t+1)$ and (t, t) (in the case of not existence of dashed edges $(t-1, t)$ and (t, t)) and in the case (d) to the vertices by metric representation $(1, 0)$ and $(1, 1)$ as well. This concludes that G is also a path or a cane. □

The above proposition shows that the inverse of Theorem 2.1 is not true. Later on, we focus on the case $k = 2$ and construct the family $\mathcal{F}$ of all 2-trees with metric dimension two.

Let $\mathcal{F}$ be the family of 2-trees, where each member G of $\mathcal{F}$ consists of a 2-tree G_0 and some branches on it that, in the case of existence, satisfying the following conditions.

(1) G_0 is a 2-path or a 2-tree that is obtained by identifying two specific edges of two disjoint 2-paths as shown in Fig. 4.

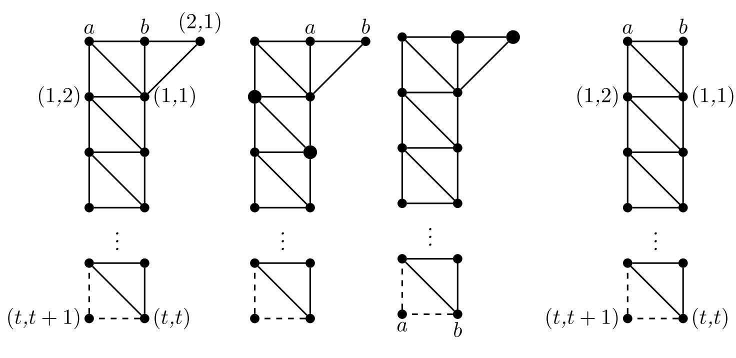

Fig. 3. The possible cases for basis $\{a, b\}$ in 2-tree G.

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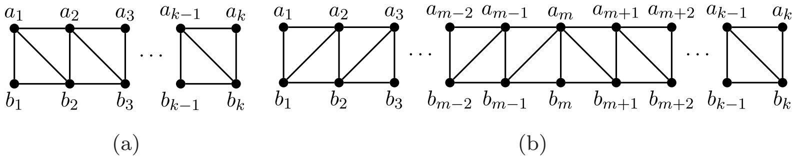

Fig. 4. Two different forms of G_0.

(2) On every edge there is at most one branch.
(3) G avoids any (a_i, a_{i+1})-branch.
(4) Each branch is either a 2-path or a cane.
(5) In each (a_i, b_i)-branch the degree of a_i is two.
(6) If G_0 is as the graph depicted in Fig. 4(b), then G avoids any (a_m, x)-branch.
(7) G contains at most one branch on the edges of the triangle containing b_ib_{i+1} in G_0.
(8) The degree of each b_i in G is at most 7.
(9) G has at most one branch of length greater than one on the edges of the triangle containing a_ia_{i+1} in G_0.
(10) If G_0 is of the form of Fig. 4(b), then (b_{m-1}, b_m)-branch and (b_m, b_{m+1})-branch are 2-paths and at most one of them is of length more than one.
(11) For every i, $2 \le i \le k-1$, at most one of the (b_{i-1}, b_i)-branches and (b_i, b_{i+1})-branches is a cane.
(12) All (a_i, b_i)-branches, (a_i, b_{i+1})-branches and (a_i, b_{i-1})-branches are 2-paths.

Theorem 2.4. *If $G \in \mathcal{F}$, then* $\dim_M(G) = 2$.

Proof. Let $G \in \mathcal{F}$. Through the proof all of notations are the same as those which are used to introduce the family $\mathcal{F}$ and G_0 in Fig. 4. Since G is not a path, $\dim_M(G) \ge 2$. Let $W = \{a_1, a_k\}$. We show in both possible cases for G_0 that W is a resolving set for G and hence, $\dim_M(G) = 2$.

Case 1. G_0 is a 2-path as shown in Fig. 4(a).
The metric representation of the vertices $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ are as follows:

$$r(a_i \,|\, W) = (i-1, k-i), \quad 1 \le i \le k,$$

$$r(b_1 \,|\, W) = (1, k),$$

$$r(b_j \,|\, W) = (j-1, k-j+1), \quad 2 \le j \le k.$$

Thus, different vertices of G_0 have different metric representations. Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(a_i \,|\, W), 1 \le i \le k\}$$
$$= \{1-k, 3-k, 5-k, \ldots, 2i-k-1, \ldots, k-3, k-1\},$$

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and

$$\begin{aligned}&\{d_1 - d_2 : (d_1, d_2) = r(b_i \,|\, W), 1 \le i \le k\}\\ &\quad = \{1-k, 2-k, 4-k, \dots, 2i-k-2, \dots, k-4, k-2\}.\end{aligned}$$

If $G = G_0$, then we are done. Suppose that $G \neq G_0$ and let H be a branch of G on an edge e of G_0. Regards to the structures of graphs in $\mathcal{F}$, we consider the following different possibilities.

- H is a branch on the vertical edge $e = a_ib_i$, $2 \le i \le k-1$.
Note that by the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \dots, x_t\}$, where $x_1 = a_i$, $x_2 = b_i$, and $E(H) = \{x_rx_s : |r-s| \le 2\}$. If j is odd, then $d(x_j, a_1) = d(x_j, a_i) + d(a_i, a_1)$ and $d(x_j, a_k) = d(x_j, a_i) + d(a_i, a_k)$. If j is even, then $d(x_j, a_1) = d(x_j, b_i) + d(b_i, a_1)$ and $d(x_j, a_k) = d(x_j, b_i) + d(b_i, a_k)$. Hence, we have

$$r(x_j \,|\, W) = \begin{cases} \left(i-1+\left\lfloor \frac{j}{2}\right\rfloor, k-i+\left\lfloor \frac{j}{2}\right\rfloor\right) & j \text{ is odd}\\ \left(i-1+\left\lfloor \frac{j}{2}\right\rfloor - 1, k-i+\left\lfloor \frac{j}{2}\right\rfloor\right) & j \text{ is even.}\end{cases}$$

  Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \le j \le t\} = \{2i-k-1, 2i-k-2\}.$$

- H is a branch on the oblique edge $e = a_ib_{i+1}$, $2 \le i \le k-1$.
By the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \dots, x_t\}$, where $x_1 = a_i$, $x_2 = b_{i+1}$, and $E(H) = \{x_rx_s : |r-s| \le 2\}$. If j is odd, then $d(x_j, a_1) = d(x_j, a_i) + d(a_i, a_1)$ and $d(x_j, a_k) = d(x_j, a_i) + d(a_i, a_k)$. If j is even, then $d(x_j, a_1) = d(x_j, b_{i+1}) + d(b_{i+1}, a_1)$ and $d(x_j, a_k) = d(x_j, b_{i+1}) + d(b_{i+1}, a_k)$. Hence, we have

$$r(x_j \,|\, W) = \begin{cases} \left(i-1+\left\lfloor \frac{j}{2}\right\rfloor, k-i+\left\lfloor \frac{j}{2}\right\rfloor\right) & j \text{ is odd}\\ \left(i-1+\left\lfloor \frac{j}{2}\right\rfloor, k-i+\left\lfloor \frac{j}{2}\right\rfloor - 1\right) & j \text{ is even.}\end{cases}$$

  Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \le j \le t\} = \{2i-k-1, 2i-k\}.$$

- H is a branch on the horizontal edge $e = b_ib_{i+1}$, $1 \le i \le k-1$.
Using the definition of $\mathcal{F}$, H is either a 2-path or a cane. Generally, assume that

$$\{x_1, x_2, \dots, x_t\} \subseteq V(H) \subseteq \{x_1, x_2, \dots, x_t\} \cup \{x\},$$

  where the induced subgraph of H on $\{x_1, x_2, \dots, x_t\}$ is a 2-path with the edge set $\{x_rx_s : |r-s| \le 2\}$. We consider two different possibilities.

(a) $x_1 = b_i$, $x_2 = b_{i+1}$. Hence, if H is a cane, then we have $N_H(x) = \{b_i, x_3\}$. Similar to the previous cases, we have

$$r(x_1 \,|\, W) = (i-1, k-i+1),$$

$$r(x_j \,|\, W) = \begin{cases} \left(i-1+\left\lfloor \frac{j}{2} \right\rfloor, k-i+\left\lfloor \frac{j}{2} \right\rfloor\right) & j \geq 3 \text{ is odd} \\ \left(i-1+\left\lfloor \frac{j}{2} \right\rfloor, k-i+\left\lfloor \frac{j}{2} \right\rfloor - 1\right) & j \text{ is even.} \end{cases}$$

Also, if H is a cane, then $r(x \,|\, W) = (i-1+1, k-i+2)$.

(b) $x_1 = b_{i+1}$, $x_2 = b_i$. Hence, if H is a cane, then we have $N_H(x) = \{b_{i+1}, x_3\}$. Similarly, we have

$$r(x_1 \,|\, W) = (i-1+1, k-i),$$

$$r(x_j \,|\, W) = \begin{cases} \left(i-1+\left\lfloor \frac{j}{2} \right\rfloor, k-i+\left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(i-1+\left\lfloor \frac{j}{2} \right\rfloor - 1, k-i+\left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is even.} \end{cases}$$

Also, if H is a cane, then $r(x \,|\, W) = (i-1+2, k-i+1)$.

Note that in both states (and regardless of being a 2-path or a cane), we have

$$\{d_1 - d_2 : (d_1, d_2) = r(v \,|\, W), v \in V(H)\} = \{2i-k-2, 2i-k-1, 2i-k\}.$$

Therefore, in all the above cases, distinct vertices of H have different metric representations. Also, the metric representations of the vertices in $V(H)$ are different from the metric representations of the vertices in $V(G_0)\backslash\{x, y\}$, where H is a (x, y)-branch. Moreover, using the subtraction value of two coordinates in the metric representation of each vertex, it is easy to check that vertices of different (possible) branches on G_0 (satisfying the conditions mentioned in the definition of $\mathcal{F}$) have different metric representations. Thus, in this case W is a resolving set for G.

Case 2. G_0 is a 2-tree of the form Fig. 4(b).
The metric representations of the vertices $\{a_1, a_2, \ldots, a_m, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_m, \ldots, b_k\}$ are as follows:

$$r(a_i \,|\, W) = (i-1, k-i), \quad 1 \leq i \leq k,$$

$$r(b_j \,|\, W) = \begin{cases} (j, k-j) & 1 \leq j \leq m-1 \\ (m, k-m+1) & j = m \\ (j-1, k-j+1) & m+1 \leq j \leq k. \end{cases}$$

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Therefore, different vertices of G_0 have different metric representations. Moreover, note that

$$\begin{aligned}\{d_1 - d_2 : (d_1, d_2) = r(a_i \,|\, W), 1 \leq i \leq k\} \\ = \{1-k, 3-k, 5-k, \ldots, 2m-k-3, 2m-k-1, 2m-k \\ +1, \ldots, k-3, k-1\},\end{aligned}$$

and

$$\begin{aligned}&\{d_1 - d_2 : (d_1, d_2) = r(b_j \,|\, W), 1 \leq j \leq k\} \\ &\quad = \{2-k, 4-k, 6-k, \ldots, 2m-k-2, 2m-k-1, 2m-k, \ldots, k-4, k-2\}.\end{aligned}$$

If $G = G_0$, then we are done. Hence, suppose that $G \neq G_0$ and let H be a branch of G on an edge e of G_0. Again, using the possible structures of H according to the definition of $\mathcal{F}$, we consider the following different cases.

- H is a branch on the vertical edge $e = a_i b_i$, $2 \leq i \leq m-1$.
  Note that by the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$, where $x_1 = a_i$, $x_2 = b_i$, and $E(H) = \{x_r x_s : |r-s| \leq 2\}$. It is straightforward to check that

$$r(x_j \,|\, W) = \begin{cases} \left(i - 1 + \left\lfloor \dfrac{j}{2} \right\rfloor, k - i + \left\lfloor \dfrac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(i + \left\lfloor \dfrac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \dfrac{j}{2} \right\rfloor - 1\right) & j \text{ is even.} \end{cases}$$

  Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \leq j \leq t\} = \{2i-k-1, 2i-k\}.$$

- H is a branch on the vertical edge $e = a_i b_i$, $m+1 \leq i \leq k-1$.
  By the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$, where $x_1 = a_i$, $x_2 = b_i$, and $E(H) = \{x_r x_s : |r-s| \leq 2\}$. We have

$$r(x_j \,|\, W) = \begin{cases} \left(i - 1 + \left\lfloor \dfrac{j}{2} \right\rfloor, k - i + \left\lfloor \dfrac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(i + \left\lfloor \dfrac{j}{2} \right\rfloor - 2, k - i + \left\lfloor \dfrac{j}{2} \right\rfloor\right) & j \text{ is even.} \end{cases}$$

  Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \leq j \leq t\} = \{2i-k-1, 2i-k-2\}.$$

- H is a branch on the oblique edge $e = a_i b_{i-1}$, $2 \leq i \leq m-1$.
  Since $G \in \mathcal{F}$, H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$,

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where $x_1 = a_i$, $x_2 = b_{i-1}$, and $E(H) = \{x_r x_s : |r - s| \leq 2\}$. We have

$$r(x_j \,|\, W) = \begin{cases} \left(i - 1 + \left\lfloor \frac{j}{2} \right\rfloor, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 2, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is even.} \end{cases}$$

Moreover,

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \leq j \leq t\} = \{2i - k - 1, 2i - k - 2\}.$$

- H is a branch on the oblique edge $e = a_i b_{i+1}$, $m + 1 \leq i \leq k - 1$.
We know that H is a 2-path and $\deg_H(a_i) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$, where $x_1 = a_i$, $x_2 = b_{i+1}$, and $E(H) = \{x_r x_s : |r - s| \leq 2\}$. Similarly, it can be easily checked that

$$r(x_j \,|\, W) = \begin{cases} \left(i - 1 + \left\lfloor \frac{j}{2} \right\rfloor, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 1\right) & j \text{ is even.} \end{cases}$$

Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \,|\, W), 1 \leq j \leq t\} = \{2i - k - 1, 2i - k\}.$$

- H is a branch on the horizontal edge $e = b_i b_{i+1}$, $1 \leq i \leq m - 2$.
Using the definition of $\mathcal{F}$, H is either a 2-path or a cane. Generally, assume that

$$\{x_1, x_2, \ldots, x_t\} \subseteq V(H) \subseteq \{x_1, x_2, \ldots, x_t\} \cup \{x\},$$

where the induced subgraph of H on $\{x_1, x_2, \ldots, x_t\}$ is a 2-path with the edge set $\{x_r x_s : |r - s| \leq 2\}$. We consider two different possibilities.

(a) $x_1 = b_i$, $x_2 = b_{i+1}$. Hence, if H is a cane, then we have $N_H(x) = \{b_i, x_3\}$. Similar to the previous cases, we have

$$r(x_1 \,|\, W) = (i, k - i),$$

$$r(x_j \,|\, W) = \begin{cases} \left(i + \left\lfloor \frac{j}{2} \right\rfloor, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 1\right) & j \geq 3 \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 2\right) & j \text{ is even.} \end{cases}$$

Also, if H is a cane, then $r(x \,|\, W) = (i + 1, k - i + 1)$.

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(b) $x_1 = b_{i+1}$, $x_2 = b_i$. Hence, if H is a cane, then we have $N_H(x) = \{b_{i+1}, x_3\}$. Similarly, we have

$$r(x_1 \mid W) = (i+1, k-i-1),$$

$$r(x_j \mid W) = \begin{cases} \left(i + \left\lfloor \frac{j}{2} \right\rfloor, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 1\right) & j \geq 3 \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 1\right) & \text{is even.} \end{cases}$$

Also, if H is a cane, then $r(x \mid W) = (i+2, k-i)$.

Note that in the both states (and regardless of being a 2-path or a cane) we have

$$\{d_1 - d_2 : (d_1, d_2) = r(v \mid W), v \in V(H)\} = \{2i-k, 2i-k+1, 2i-k+2\}.$$

- H is a branch on the horizontal edge $e = b_{m-1}b_m$.
By the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(b_{m-1}) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$, where $x_1 = b_{m-1}$, $x_2 = b_m$, and $E(H) = \{x_r x_s : |r-s| \leq 2\}$. We have

$$r(x_j \mid W) = \begin{cases} \left(m + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - m + \left\lfloor \frac{j}{2} \right\rfloor + 1\right) & j \text{ is odd} \\ \left(m + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - m + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is even.} \end{cases}$$

Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \mid W), 1 \leq j \leq t\} = \{2m-k-2, 2m-k-1\}.$$

- H is a branch on the horizontal edge $e = b_m b_{m+1}$.
By the definition of $\mathcal{F}$, H is a 2-path and $\deg_H(b_{m+1}) = 2$. Let $V(H) = \{x_1, x_2, \ldots, x_t\}$, where $x_1 = b_{m+1}$, $x_2 = b_m$, and $E(H) = \{x_r x_s : |r-s| \leq 2\}$. We have

$$r(x_j \mid W) = \begin{cases} \left(m + \left\lfloor \frac{j}{2} \right\rfloor, k - m + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is odd} \\ \left(m + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - m + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ even.} \end{cases}$$

Moreover, note that

$$\{d_1 - d_2 : (d_1, d_2) = r(x_j \mid W), 1 \leq j \leq t\} = \{2m-k-1, 2m-k\}.$$

- H is a branch on the horizontal edge $e = b_i b_{i+1}$, $m+1 \leq i \leq k-1$.
Using the definition of $\mathcal{F}$, H is either a 2-path or a cane. Generally, assume that

$$\{x_1, x_2, \ldots, x_t\} \subseteq V(H) \subseteq \{x_1, x_2, \ldots, x_t\} \cup \{x\},$$

where the induced subgraph of H on $\{x_1, x_2, \ldots, x_t\}$ is a 2-path with the edge set $\{x_r x_s : |r-s| \leq 2\}$. Again, we consider two different possibilities.

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(a) $x_1 = b_i$, $x_2 = b_{i+1}$. Hence, if H is a cane and $N_H(x) = \{b_i, x_3\}$, then we have

$$r(x_1 \,|\, W) = (i-1, k-i+1),$$
$$r(x_j \,|\, W) = \begin{cases} \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \geq 3 \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \frac{j}{2} \right\rfloor - 1\right) & j \text{ is even.} \end{cases}$$

Also, if H is a cane, then $r(x \,|\, W) = (i, k-i+2)$.

(b) $x_1 = b_{i+1}$, $x_2 = b_i$. Hence, if H is a cane, then we have $N_H(x) = \{b_{i+1}, x_3\}$. Similarly, we have

$$r(x_1 \,|\, W) = (i, k-i),$$
$$r(x_j \,|\, W) = \begin{cases} \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 1, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \geq 3 \text{ is odd} \\ \left(i + \left\lfloor \frac{j}{2} \right\rfloor - 2, k - i + \left\lfloor \frac{j}{2} \right\rfloor\right) & j \text{ is even.} \end{cases}$$

Also, if H is a cane, then $r(x \,|\, W) = (i+1, k-i+1)$.

Note that in the both states (and regardless of being a 2-path or a cane) we have

$$\{d_1 - d_2 : (d_1, d_2) = r(v \,|\, W), v \in V(H)\} = \{2i-k-2, 2i-k-1, 2i-k\}.$$

Therefore, in all of above cases, distinct vertices of H have different metric representations. Also, the metric representations of the vertices in $V(H)$ are different from the metric representations of the vertices in $V(G_0)\backslash\{x, y\}$, where H is a (x, y)-branch. Moreover, using the subtraction value of two coordinates in the metric representation of each vertex, it is easy to check that vertices of different (possible) branches on G_0 (satisfying the conditions mentioned in the definition of $\mathcal{F}$) have different metric representations. Thus, in this case W is a resolving set for G. □

To prove the converse of Theorem 2.4, we need the following lemma.

Lemma 2.5. *Let H be a $\{u, v\}$-branch of G and let $\{a, b\}$ be a basis for $G \cup H$. If $\{a, b\} \cap V(H) \subseteq \{u, v\}$, then $\{u, v\}$ is a metric basis for H.*

Proof. Suppose on the contrary, there are two different vertices x and y in H such that

$$d(x, u) = d(y, u) = r, \quad d(x, v) = d(y, v) = s.$$

Since H is a branch on $\{u, v\}$, each path connecting a vertex in H with a vertex in $V(G)\backslash V(H)$ passes through u or v. Assume that

$$d(u, a) = r_1, \quad d(v, a) = s_1, \quad d(u, b) = r_2, \quad d(v, b) = s_2.$$

Hence,

$$d(x,a) = \min\{r + r_1, s + s_1\} = d(y,a), \quad d(x,b) = \min\{r + r_2, s + s_2\} = d(y,b).$$

This contradicts that $\{a, b\}$ is a resolving set for $G \cup H$. □

Now, we prove that every two-dimensional 2-tree belongs to the family $\mathcal{F}$.

Theorem 2.6. *If G is a 2-tree of metric dimension two, then $G \in \mathcal{F}$.*

Proof. Let G be a 2-tree and $\{a, b\}$ be a basis of G. If $d(a, b) = 1$, then by Proposition 2.3, G is a 2-path or a cane which belongs to $\mathcal{F}$. Thus, assume that $d(a, b) > 1$ and let H be a minimal induced 2-connected subgraph of G as shown in Fig. 5, containing a and b. Since the clique number of G is three, in each square exactly one of the dashed edges are allowed. Moreover, by the minimality of H, we have $\deg_H(a) = \deg_H(b) = 2$, where $a \in \{a_1, b_1\}$ and $b \in \{a_k, b_k\}$. Hence, one of two vertices a_1, b_1 or one of two vertices a_k, b_k may not exist. One can check that $\{a, b\} \neq \{a_1, b_k\}$ and $\{a, b\} \neq \{b_1, a_k\}$, otherwise, two neighbors of a or b get the same metric representation. Thus, by the symmetry, we may assume $\{a, b\} = \{a_1, a_k\}$.

If $\Delta(H) \leq 4$, then H is a 2-path as shown in Fig. 4(a). Otherwise $\Delta(H) = 5$. If there exists a vertex b_j of degree 5, then it can be easily checked that b_j and a_j have the same representation with respect to $\{a_1, a_k\}$. Also, existence of two vertices a_i and $a_{i'}$ both of degree 5, $i \leq i'$, implies that there exists some vertex b_j, $i \leq j \leq i'$, of degree 5, which is impossible. Thus, there exists a unique a_i of degree 5. Therefore, H is the graph shown in Fig. 4(b). Thus, H is a 2-path or a 2-tree obtained by identifying the specific edge, say $a_m b_m$, of two 2-paths (see Fig. 4(b)), where $B = \{a_1, a_k\}$. Thus, G satisfies property (1).

Clearly, on every edge there is at most one branch; thus, property (2) follows. Also, G avoids any (a_i, a_{i+1})-branch, because each vertex adjacent to both a_i and a_{i+1} has the same metric representation as b_i or b_{i+1}. Thus, G contains only (a_i, b_i)-branches, (a_i, b_{i+1})-branches, (a_{i+1}, b_i)-branches or (b_i, b_{i+1})-branches; which implies property (3). Moreover, by Proposition 2.3 and Lemma 2.5, each of these branches is a 2-path or a cane. Therefore, property (4) holds. Also, by Theorem 1.1, for every i, $1 \leq i \leq k$, there is at most one (a_i, x)-branch in G. Moreover, in each (a_i, b_i)-branch the degree of a_i is two, which shows trueness of property (5).

To see property (6), first note that by property (3) there is no (a_{m-1}, a_m)-branch or (a_m, a_{m+1})-branch. Moreover, in each (a_m, x)-branch, for $x \in \{b_{m-1}, b_m, b_{m+1}\}$, the unique neighbor of a_m on the branch has the same metric representation as b_m.

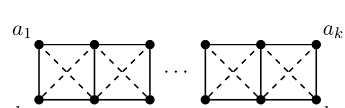

Fig. 5. A minimal induced 2-connected subgraph of G.

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To show that G has property (7), suppose that a triangle $a_ib_ib_{i+1}$ has more than one branch. By Theorem 1.1, at most one of (a_i, b_i)-branch and (a_i, b_{i+1})-branch exists. Therefore, b_ib_{i+1} has a branch H_1 and one of the edges a_ib_i or a_ib_{i+1} has another branch H_2. Let x and y be the vertices of distance one from G_0 on branches H_1 and H_2, respectively. Hence, $d(a_1, x) = d(a_1, y) = i$ and $d(a_k, x) = d(a_k, y) = k - i + 1$. That is, $\{a_1, a_k\}$ is not a basis of G, which is a contradiction. A similar reason works for triangle $a_ib_{i-1}b_i$. Hence, G has property (7).

Let (d_1, d_2) be metric representation of b_i. Then metric representations of each neighbor of b_i which is out of G_0 could be one of $(d_1 + 1, d_2 + 1), (d_1 + 1, d_2)$ or $(d_1, d_2 + 1)$. Thus, b_i has at most three neighbors out of G_0. Hence, the degree of b_i in G is at most 7 that is property (8).

If there are two branches of length at least 2 on a triangle containing a_ia_{i+1}, then the metric representation of the second vertices on these branches are the same, a contradiction. Thus, G satisfies property (9).

If H is a (b_{m-1}, b_m)-branch of cane type, then one can find two vertices in $N_G(b_m) \cup N_G(b_{m-1})$ with the same metric representation. A similar argument holds whenever H is a (b_m, b_{m+1})-branch of cane type. If there is a (b_{m-1}, b_m)-branch, say H_1, and a (b_m, b_{m+1})-branch, say H_2, both of length at least two, then b_m has a neighbor in H_1 with the same metric representation as a neighbor of b_m in H_2. Hence, property (10) holds.

Suppose that two branches on (b_{i-1}, b_i) and (b_i, b_{i+1}) are canes. In this case, it can be checked that in the set of neighbors of b_i in these branches there are two vertices with the same metric representation. Thus, G satisfies property (11).

Using Theorem 1.1, the degree of each a_i in G, $1 < i < n$, is at most five. Note that $\deg(a_i) \in \{4, 5\}$. Now suppose that H is a branch on the edge $\{a_i, b_i\}$, $\{a_i, b_{i+1}\}$ or $\{a_i, b_{i-1}\}$. If H is a cane, then $\deg_G(a_i) \geq 6$ or two neighbors of b_{i-1}, b_i or b_{i+1} in H get the same metric representation, which both are contradictions. Thus, each branch on the edge $\{a_i, b_{i-1}\}$, $\{a_i, b_i\}$ or $\{a_i, b_{i+1}\}$ is a 2-path and G satisfies property (12). □

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