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On randomly k-dimensional graphs

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ABSTRACT

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set for G with minimum cardinality is called a basis of G and its cardinality is the metric dimension of G. A connected graph G is called a randomly k-dimensional graph if each k-set of vertices of G is a basis of G. In this work, we study randomly k-dimensional graphs and provide some properties of these graphs.

1. Introduction

We refer the reader to [1] for graphical notation and terminology not described in this work. Throughout the work, $G = (V, E)$ is a finite, simple, and connected graph. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. Also, $N(v)$ is the set of all neighbors of vertex v and $\deg(v) = |N(v)|$ is the degree of vertex v. The maximum degree of the graph G, $\Delta(G)$, is $\max_{v\in V(G)} \deg(v)$. We mean by $\omega(G)$ the number of vertices in a maximum clique in G. For a subset S of $V(G)$, $G \setminus S$ is the subgraph $\langle V(G) \setminus S\rangle$ induced by $V(G) \setminus S$ in G. A set $S \subseteq V(G)$ is a separating set in G if $G \setminus S$ has at least two connected components. We call a vertex $v \in V(G)$ a cut vertex of G if $\{v\}$ is a separating set in G. If $G \neq K_n$ has no cut vertex, then G is called a 2-connected graph. $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relations between u and v, respectively. The symbol $(v_1, v_2, \ldots, v_n)$ represents a path of order n, P_n.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the (*metric*) *representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. A resolving set for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$.

For example, the graphs G and H in Fig. 1 have the basis $B = \{v_1, v_2\}$ and hence $\beta(G) = \beta(H) = 2$. The representations of vertices of G with respect to B are

$$r(v_1|B) = (0, 1), \quad r(v_2|B) = (1, 0), \quad r(v_3|B) = (2, 1), \quad r(v_4|B) = (2, 2), \quad r(v_5|B) = (1, 2).$$

Also, the representations of vertices of H with respect to B are

$$r(v_1|B) = (0, 1), \quad r(v_2|B) = (1, 0), \quad r(v_3|B) = (1, 1), \quad r(v_4|B) = (2, 2), \quad r(v_5|B) = (1, 2).$$

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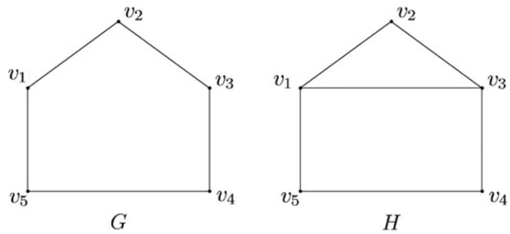

Fig. 1. $\mathrm{bas}(G) = \beta(G) = \mathrm{res}(G)$ and $\mathrm{bas}(H) \neq \beta(H) \neq \mathrm{res}(H)$.

To see whether a given set W is a resolving set for G, it is sufficient to look at the representations of vertices in $V(G) \setminus W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$. When W is a resolving set for G, we say that W *resolves* G. In general, we say that an ordered set W resolves a set T of vertices in G if the representations of vertices in T are distinct with respect to W. When $W = \{x\}$, we say that vertex x resolves T.

In [2], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for what we call a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with US Sonar and Coast Guard Loran stations. Independently, Harary and Melter [3] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [4–8]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [9], network discovery and verification [10], robot navigation [7], the mastermind game [4], problems of pattern recognition and image processing [11], and combinatorial search and optimization [9].

The following simple result is very useful.

Observation 1 (*[12]*)**.** Let G be a graph and $u, v \in V(G)$ such that $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. If W resolves G, then u or v is in W.

It is obvious that for a graph G of order n, $1 \le \beta(G) \le n - 1$.

Theorem A (*[13]*)**.** *Let G be a graph of order n. Then,*

(i) $\beta(G) = 1$ *if and only if* $G = P_n$,
(ii) $\beta(G) = n - 1$ *if and only if* $G = K_n$.

The *basis number*, $\mathrm{bas}(G)$, of G is the maximum integer r such that every r-set of vertices of G is a subset of some basis of G. Also, the *resolving number*, $\mathrm{res}(G)$, of G is the minimum integer k such that every k-set of vertices of G is a resolving set for G. These parameters are introduced in [14,15], respectively. Clearly, if G is a graph of order n, then $0 \le \mathrm{bas}(G) \le \beta(G)$ and $\beta(G) \le \mathrm{res}(G) \le n - 1$. Chartrand et al. in [14] considered graphs G with $\mathrm{bas}(G) = \beta(G)$. They called these graphs *randomly k-dimensional* graphs, where $k = \beta(G)$. Obviously, $\mathrm{bas}(G) = \beta(G)$ if and only if $\mathrm{res}(G) = \beta(G)$. In other words, a randomly k-dimensional graph is a graph for which every k-set of its vertices is a basis. For example in graph G of Fig. 1, if W is a set of two adjacent vertices, then the representations of vertices in $V(G) \setminus W$ with respect to W are $(1, 2)$, $(2, 2)$, and $(2, 1)$. Also, if W is a set of two non-adjacent vertices, then the representations of vertices in $V(G) \setminus W$ with respect to W are $(1, 1)$, $(1, 2)$, and $(2, 1)$. Therefore, G is a randomly two-dimensional graph. But, in graph H of Fig. 1, $\{v_1, v_4\}$ is not a resolving set; hence H is not a randomly two-dimensional graph. Since $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_4, v_5\}$ are bases of H, $\mathrm{bas}(H) = 1$. Also, $\mathrm{res}(H) = 3$, because every 3-set of $V(H)$ is a resolving set in H.

Obviously, K_1 and K_2 are the only randomly one-dimensional graphs. Chartrand et al. [14] proved that a graph G is randomly two-dimensional if and only if G is an odd cycle. In this work, we first characterize all graphs of order n and resolving number 1 and $n - 1$. Then, we provide some properties of randomly k-dimensional graphs.

2. The main results

We first characterize all graphs G with $\mathrm{res}(G) = 1$ and all graphs G of order n with $\mathrm{res}(G) = n - 1$.

Theorem 1. *Let G be a graph of order n. Then,*

(i) $\mathrm{res}(G) = 1$ *if and only if* $G \in \{P_1, P_2\}$,
(ii) $\mathrm{res}(G) = n - 1$ *if and only if* $N(v) \setminus \{u\} = N(u) \setminus \{v\}$, *for some* $u, v \in V(G)$.

Proof. (i) It is easy to see that for $G \in \{P_1, P_2\}$, $\mathrm{res}(G) = 1$. Conversely, suppose that $\mathrm{res}(G) = 1$. Thus, $1 \le \beta(G) \le \mathrm{res}(G) = 1$ and hence, $\beta(G) = 1$. Therefore, by Theorem A, $G = P_n$. If $n \ge 3$, then P_n has a vertex of degree 2 and this vertex does not resolve its neighbors. Thus, $\mathrm{res}(G) \ge 2$, which is a contradiction. Consequently, $n \le 2$, that is $G \in \{P_1, P_2\}$.

(ii) Let $u, v \in V(G)$ be such that $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. If $\mathrm{res}(G) \le n - 2$, then the set $V(G) \setminus \{u, v\}$ is a resolving set for G. But, by Observation 1, every resolving set for G contains at least one of the vertices u and v. This contradiction implies that $\mathrm{res}(G) = n-1$. Conversely, suppose that $\mathrm{res}(G) = n-1$. Thus, there exists a subset T of $V(G)$ with cardinality $n-2$ such that T is not a resolving set for G. Assume that $T = V(G) \setminus \{u, v\}$. If $N(u) \setminus \{v\} \neq N(v) \setminus \{u\}$, then there exists a vertex $w \in T$ which is adjacent to only one of the vertices u and v and hence, $d(u, w) \neq d(v, w)$. Since $w \in T$, T resolves G, which is a contradiction. Therefore, $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. □

Corollary 1. *If $G \neq K_n$ is a randomly k-dimensional graph, then for each pair of vertices $u, v \in V(G)$, $N(v) \setminus \{u\} \neq N(u) \setminus \{v\}$.*

Proof. If $N(v) \setminus \{u\} = N(u) \setminus \{v\}$ for some $u, v \in V(G)$, then by Theorem 1, $\mathrm{res}(G) = n-1$, where n is the order of G. Since G is a randomly k-dimensional graph, $\beta(G) = \mathrm{res}(G) = n-1$. Therefore, by Theorem A, $G = K_n$, which is a contradiction. Hence, for each $u, v \in V(G)$, $N(v) \setminus \{u\} \neq N(u) \setminus \{v\}$. □

Lemma 1. *If G is a randomly k-dimensional graph with $k \geq 2$ and minimum degree δ, then $\delta \geq 2$.*

Proof. Suppose on the contrary that there exists a vertex $u \in V(G)$ with $\deg(u) = 1$. Let v be the unique neighbor of u and $T \subseteq V(G)$ be a subset of $V(G)$ with $|T| = k$ and $u, v \in T$. Since G is a randomly k-dimensional graph, $T \setminus \{v\}$ is not a resolving set for G. Thus, there exists a pair of vertices $x, y \in V(G)$ such that $d(x, v) \neq d(y, v)$ and $d(x, t) = d(y, t)$ for each $t \in T \setminus \{v\}$. Hence, $d(x, u) = d(y, u)$. Clearly, if $u \in \{x, y\}$, then $d(x, u) \neq d(y, u)$, which is a contradiction. Consequently, $u \notin \{x, y\}$. Therefore, $d(x, u) = d(x, v) + 1$ and $d(y, u) = d(y, v) + 1$. Thus, $d(x, v) = d(y, v)$. This contradiction implies that $\delta \geq 2$. □

Theorem 2. *If $k \geq 2$, then every randomly k-dimensional graph is 2-connected.*

Proof. Suppose on the contrary that u is a cut vertex in G. Let G_1 be a connected component of $G \setminus \{u\}$. Set $H_2 := G \setminus V(G_1)$ and $H_1 := \langle V(G_1) \cup \{u\} \rangle$, the induced subgraph by $V(G_1) \cup \{u\}$ of G. Note that for each $x \in V(H_1)$ and each $y \in V(H_2)$, $d(x, y) = d(x, u) + d(u, y)$. By Lemma 1, G does not have any vertex of degree 1. Therefore, $|V(H_1)| \geq 3$ and $|V(H_2)| \geq 3$. Suppose that $a, b \in V(H_2)$ and $V(H_1)$ resolves $\{a, b\}$. Then, there exists a vertex $w \in V(H_1)$ such that $d(a, w) \neq d(b, w)$. Thus, $d(a, u) + d(u, w) \neq d(b, u) + d(u, w)$, that is $d(a, u) \neq d(b, u)$. Hence, $V(H_1)$ resolves a pair of vertices of $V(H_2)$ if and only if u resolves this pair. If $V(H_1)$ is a resolving set for G, then $\{u\}$ is a resolving set for H_2. Therefore, by Theorem A, H_2 is a path. Since $|V(H_2)| \geq 3$, G has a vertex of degree 1, which contradicts Lemma 1. Hence, $\beta(H_2) \geq 2$ and $V(H_1)$ does not resolve G. Now, one of the following two cases can happen.

1. u belongs to a basis of H_2. In this case u along with $\beta(H_2) - 1$ vertices of $V(H_2) \setminus \{u\}$ forms a basis T of H_2. Since $\beta(H_2) \geq 2$, there exists a vertex $x \in T \setminus \{u\}$. Note that $T \cup V(H_1) \setminus \{x\}$ is not a resolving set for G; otherwise $T \setminus \{x\}$ is a resolving set for H_2 of size $\beta(H_2) - 1$. Thus,

$$\mathrm{res}(G) \geq |T \cup V(H_1)| = \beta(H_2) + |V(H_1)| - 1.$$

Now, suppose that $z \in V(G_1)$. Since $|V(H_1)| \geq 3$ and G_1 is a connected component of $G \setminus \{u\}$, z has a neighbor in G_1, say v. Therefore, $d(z, v) = 1 \neq d(y, v)$ for each $y \in V(H_2) \setminus \{u\}$. Hence, the set $T \cup V(H_1) \setminus \{z\}$ is a resolving set for G. Thus,

$$\beta(G) \leq |T \cup V(H_1) \setminus \{z\}| = \beta(H_2) + |V(H_1)| - 2.$$

Consequently, $\beta(G) < \mathrm{res}(G)$, which is a contradiction.

2. u does not belong to any basis of H_2. Let T be a basis of G and $x \in T$. Therefore, $T \cup V(H_1) \setminus \{x\}$ is not a resolving set for G. Hence,

$$\mathrm{res}(G) \geq |T \cup V(H_1)| = \beta(H_2) + |V(H_1)|.$$

Now, suppose that $z \in V(G_1)$. Like in the previous case, $T \cup V(H_1) \setminus \{z\}$ is a resolving set for G. Thus,

$$\beta(G) \leq |T \cup V(H_1) \setminus \{z\}| = \beta(H_2) + |V(H_1)| - 1.$$

Therefore, $\beta(G) < \mathrm{res}(G)$, which is impossible.

Consequently, G does not have any cut vertex. □

Theorem 3. *If G is a randomly k-dimensional graph with $k \geq 4$, then there are no adjacent vertices of degree 2 in G.*

Proof. Suppose on the contrary that G has adjacent vertices of degree 2. Therefore, there is an induced subgraph $P_r = (a_1, a_2, \ldots, a_r)$, $r \geq 2$, such that for each i, $1 \leq i \leq r$, $\deg(a_i) = 2$ in G. Suppose that $x, y \in V(G) \setminus V(P_r)$ and $x \sim a_1$, $y \sim a_r$. Since $k \geq 4$, G is not a cycle. Thus, Theorem 2 implies that $x \neq y$; otherwise, $x = y$ is a cut vertex in G. By assumption, G has a basis $B = \{x, y, a_i, a_j\} \cup T$, where $1 \leq i \neq j \leq r$ and T is a subset of $V(G) \setminus \{x, y, a_i, a_j\}$ with $|T| = k - 4$. Now, one of the following cases can happen.

1. r is odd. Suppose that $B_1 = B \cup \left\{a_{\frac{r+1}{2}}\right\} \setminus \{a_i, a_j\}$. We claim that B_1 is a resolving set for G. Otherwise, there exist vertices $u, v \in V(G)$ with $r(u|B_1) = r(v|B_1)$. If $v \in V(P_r)$ and $u \notin V(P_r)$, then $d\left(v, a_{\frac{r+1}{2}}\right) \leq \frac{r-1}{2}$ and $d\left(u, a_{\frac{r+1}{2}}\right) \geq \frac{r+1}{2}$. Hence, $r(u|B_1) \neq r(v|B_1)$, which is a contradiction. Therefore, both of the vertices u and v belong to $V(P_r)$ or $V(G) \setminus V(P_r)$. If $u, v \in V(P_r)$, then $d\left(u, a_{\frac{r+1}{2}}\right) = d\left(v, a_{\frac{r+1}{2}}\right)$ implies $u, v \in \left\{a_{\frac{r+1}{2}-i}, a_{\frac{r+1}{2}+i}\right\}$ for some i, $1 \leq i \leq \frac{r-1}{2}$. On the other

hand, $d\left(x, a_{\frac{r+1}{2}-i}\right) = \frac{r+1}{2} - i$ and $d\left(x, a_{\frac{r+1}{2}+i}\right) = \min\left\{\frac{r+1}{2}+i, \frac{r+1}{2}-i+d(x,y)\right\}$. If $\frac{r+1}{2}+i \le \frac{r+1}{2}-i+d(x,y)$, then $d\left(x, a_{\frac{r+1}{2}-i}\right) \neq d\left(x, a_{\frac{r+1}{2}+i}\right)$, which is a contradiction. Thus, $\frac{r+1}{2}-i+d(x,y) < \frac{r+1}{2}+i$ and hence, $\frac{r+1}{2}-i+d(x,y) = \frac{r+1}{2}-i$, because $d\left(x, a_{\frac{r+1}{2}-i}\right) = d\left(x, a_{\frac{r+1}{2}+i}\right)$. Therefore, $d(x,y) = 0$, which contradicts $x \neq y$. Thus, $u, v \in V(G) \setminus V(P_r)$. Since $r(u|B_1) = r(v|B_1)$ and B is a resolving set for G, there exists a vertex in $B \setminus B_1 = \{a_i, a_j\} \setminus \left\{a_{\frac{r+1}{2}}\right\}$ which resolves $\{u, v\}$. By symmetry, we can assume that a_i resolves $\{u, v\}$. Therefore, $d(u, a_i) \neq d(v, a_i)$, $d(u, x) = d(v, x)$, and $d(u, y) = d(v, y)$. But,

$$d(u, a_i) = \min\{d(u, x) + d(x, a_i), d(u, y) + d(y, a_i)\},$$

and

$$d(v, a_i) = \min\{d(v, x) + d(x, a_i), d(v, y) + d(y, a_i)\}.$$

If $d(u,x)+d(x,a_i) \le d(u,y)+d(y,a_i)$ and $d(v,x)+d(x,a_i) \le d(v,y)+d(y,a_i)$, then $d(u,x)+d(x,a_i) \neq d(v,x)+d(x,a_i)$, which implies $d(u,x) \neq d(v,x)$, a contradiction. Similarly, if $d(u,y) + d(y,a_i) \le d(u,x) + d(x,a_i)$ and $d(v,y) + d(y,a_i) \le d(v,x) + d(x,a_i)$, then $d(u,y) \neq d(v,y)$, which is a contradiction. Therefore, by symmetry, we can assume that $d(u,x) + d(x,a_i) \le d(u,y) + d(y,a_i)$ and $d(v,y) + d(y,a_i) \le d(v,x) + d(x,a_i)$. Thus,

$$d(u, a_i) = d(u, x) + d(x, a_i) = d(v, x) + d(x, a_i) \ge d(v, a_i),$$

and

$$d(v, a_i) = d(v, y) + d(y, a_i) = d(u, y) + d(y, a_i) \ge d(u, a_i).$$

These imply that $d(u, a_i) = d(v, a_i)$, which is a contradiction. Therefore, B_1 is a resolving set for G with cardinality $k - 1$.

2. r is even. Suppose that $B_2 = B \cup \left\{a_{\frac{r}{2}}\right\} \setminus \{a_i, a_j\}$. Like in the previous case, B_2 is a resolving set for G with cardinality $k - 1$.

In both cases, we get a contradiction to the assumption that G is a randomly k-dimensional graph. Therefore, there are no adjacent vertices of degree 2 in G. □

Theorem 4. *If G is a randomly k-dimensional graph and T is a separating set of G with $|T| = k - 1$, then $G \setminus T$ has exactly two connected components and for each pair of vertices $u, v \in V(G) \setminus T$ with $r(u|T) = r(v|T)$, u and v belong to different components.*

Proof. Since $\beta(G) = k$ and $|T| = k - 1$, there exist two vertices $u, v \in V(G) \setminus T$ with $r(u|T) = r(v|T)$. Let H be a connected component of $G \setminus T$ for which $u \notin H$ and $v \notin H$. If $w \in H$, then there exist two vertices $s, t \in T$ such that $d(u, w) = d(u, s) + d(s, w)$ and $d(v, w) = d(v, t) + d(t, w)$. Since $r(u|T) = r(v|T)$, we have $d(u, s) = d(v, s)$ and $d(u, t) = d(v, t)$. Therefore,

$$d(u, w) = d(u, s) + d(s, w) = d(v, s) + d(s, w) \ge d(v, w).$$

Also,

$$d(v, w) = d(v, t) + d(t, w) = d(u, t) + d(t, w) \ge d(u, w).$$

Hence, $d(u, w) = d(v, w)$. Thus, $r(u|T \cup \{w\}) = r(v|T \cup \{w\})$. Consequently, $T \cup \{w\}$ is not a resolving set for G and $|T \cup \{w\}| = k$. This contradicts the assumption that G is randomly k-dimensional. Therefore, $G \setminus T$ has exactly two components and u and v belong to different components. □

Corollary 2. *If G is a randomly k-dimensional graph with $k \ge 2$, then $\Delta(G) \ge k$.*

Proof. If $G = K_n$, then $\Delta(G) = n - 1 = k$. Now suppose that $G \neq K_n$, and suppose on the contrary that $\Delta(G) \le k - 1$. Suppose that $u \in V(G)$, $\deg(u) = \Delta(G)$, and let T be a subset of $V(G)$ with $|T| = k - 1$ and $N(u) \subseteq T$. By Theorem 4, $G \setminus T$ has exactly two connected components, of which one is $\{u\}$. Since $|T| = k - 1$ and $\beta(G) = k$, there exist two vertices $x, y \in V(G) \setminus T$ such that $r(x|T) = r(y|T)$. By Theorem 4, x and y belong to different components. Therefore, one of them is u, say $x = u$. Since $r(u|T) = r(y|T)$, we have $N(u) \subseteq N(y)$. By Corollary 1, G does not have any pair of vertices u, v with $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Hence, $N(u) \subset N(y)$; this contradicts $\deg(u) = \Delta(G)$. Therefore, $\Delta(G) \ge k$. □

Corollary 3. *If u and v are two non-adjacent vertices in a randomly k-dimensional graph, then* $\deg(u) + \deg(v) \ge k$.

Proof. If $|N(u) \cup N(v)| \le k - 1$, then let T be a subset of $V(G) \setminus \{u, v\}$ with $|T| = k - 1$ and $N(u) \cup N(v) \subseteq T$. By Theorem 4, $G \setminus T$ has exactly two connected components $\{u\}$ and $\{v\}$. Hence, $|T| = n - 2$. This implies that $k = n - 1$ and by Theorem 1, $G = K_n$. Consequently, $u \sim v$, which is a contradiction. Thus, $\deg(u) + \deg(v) \ge |N(u) \cup N(v)| \ge k$. □

Theorem 5. *If G is a randomly k-dimensional graph of order at least 2, then $\omega(G) \le k + 1$. Moreover, $\omega(G) = k + 1$ if and only if $G = K_n$.*

Proof. Let H be a clique of size $\omega(G)$ in G and T be a subset of $V(H)$ with $|T| = \omega(G) - 2$. If $T = V(H) \setminus \{u, v\}$, then $r(u|T) = (1, 1, \dots, 1) = r(v|T)$. Therefore, T is not a resolving set for G. Since G is a randomly k-dimensional graph, $|T| \leq k - 1$. Thus, $\omega(G) - 2 = |T| \leq k - 1$. Consequently, $\omega(G) \leq k + 1$.

Clearly, if $G = K_n$, then $\omega(G) = k + 1$. Conversely, suppose that $\omega(G) = k + 1$. If $G \neq K_n$, then there exists a vertex $x \in V(G) \setminus V(H)$ such that x is adjacent to some vertices of $V(H)$, because G is connected. Since $|V(H)| = \omega(G)$, x is not adjacent to all vertices of $V(H)$. If there exist vertices $y, z \in V(H)$ such that $y \nsim x$ and $z \nsim x$, then $d(x, y) = d(x, z) = 2$, because x is adjacent to some vertices of H. Suppose that $S = \{x\} \cup V(H) \setminus \{y, z\}$. Therefore, $r(y|S) = (2, 1, 1, \dots, 1) = r(z|S)$. Thus, S is not a resolving set for G and $|S| = k$, which is a contradiction. Hence, x is adjacent to $\omega(G) - 1$ vertices of H.

On the other hand, x is adjacent to at most one vertex of H. Otherwise, there exist vertices $s, t \in V(H)$ such that $s \sim x$ and $t \sim x$. Suppose that $R = \{x\} \cup V(H) \setminus \{s, t\}$. Therefore, $r(s|R) = (1, 1, \dots, 1) = r(t|R)$. Thus, R is not a resolving set for G and $|R| = k$, which is a contradiction. Consequently, $\omega(G) = 2$ and $k = \omega(G) - 1 = 1$. Therefore, $G = K_2$, which contradicts $G \neq K_n$. Hence, $G = K_n$. □

Lemma 2. *If* $\mathrm{res}(G) = k$*, then each two vertices of* G *have at most* $k - 1$ *common neighbors.*

Proof. Suppose that $u, v \in V(G)$ and $T = N(u) \cap N(v)$. Thus, $r(u|T) = (1, 1, \dots, 1) = r(v|T)$. Therefore, T is not a resolving set for G. Since G is a randomly k-dimensional graph, $|N(u) \cap N(v)| = |T| \leq k - 1$. □

Theorem 6. *If* $G \neq K_n$ *is a randomly k-dimensional graph of order n, then* $\Delta(G) \leq n - 2$.

Proof. Suppose on the contrary that there exists a vertex $u \in V(G)$ with $\deg(u) = n - 1$. For each $T \subseteq V(G) \setminus \{u\}$ with $|T| = k - 1$, the set $T \cup \{u\}$ is a resolving set for G while T is not a resolving set for G. Hence, there exist vertices $x, y \in V(G) \setminus T$ such that $r(x|T) = r(y|T)$ and $d(x, u) \neq d(y, u)$. Since u is adjacent to all vertices of G, we have $u \in \{x, y\}$, say $x = u$. Thus, $r(y|T) = r(u|T) = (1, 1, \dots, 1)$. By Lemma 2, $|N(u) \cap N(y)| \leq k - 1$. Hence, $\deg(y) \leq k$, because u is adjacent to all vertices of G. This gives $N(y) = T \cup \{u\}$.

Now, suppose that $S = T \cup \{y\} \setminus \{v\}$, for an arbitrary vertex $v \in T$. Since $|S| = k - 1$, S is not a resolving set for G. Therefore, there exist vertices $a, b \in V(G) \setminus S$ such that $r(a|S) = r(b|S)$. Since $S \cup \{u\}$ is a resolving set for G, we have $d(a, u) \neq d(b, u)$. Hence, $u \in \{a, b\}$, say $b = u$. Thus, $r(a|S) = r(u|S) = (1, 1, \dots, 1)$. Consequently, $a \sim y$. Therefore, $a \in T$, because $N(y) = T \cup \{u\}$ and $a \neq u$. Hence, $a \in (V(G) \setminus S) \cap T = \{v\}$, that is $a = v$. Thus, v is adjacent to all vertices of $T \setminus \{v\}$. Since v is an arbitrary vertex of T, T is a clique. Therefore, $T \cup \{u, y\}$ is a clique of size $k + 1$ in G. Consequently, by Theorem 5, $G = K_n$, which is a contradiction. Thus, $\Delta(G) \leq n - 2$. □

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