Local Clique Covering of Claw-free Graphs

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Abstract

A $k-$*clique covering* of a simple graph G is a collection of cliques of G covering all the edges of G such that each vertex is contained in at most k cliques. The smallest k for which G admits a $k-$clique covering is called the local clique cover number of G and is denoted by $\mathrm{lcc}(G)$. Local clique cover number can be viewed as the local counterpart of the clique cover number which is equal to the minimum total number of cliques covering all edges. In this paper, several aspects of the local clique covering problem are studied and its relationships to other well-known problems are discussed. In particular, it is proved that the local clique cover number of every claw-free graph is at most $c\Delta/\log\Delta$, where Δ is the maximum degree of the graph and c is a constant. It is also shown that the bound is tight, up to a constant factor. Moreover, regarding a conjecture by Chen et al. (Clique covering the edges of a locally cobipartite graph, Discrete Math., 219(1-3):17–26, 2000), we prove that the clique cover number of every connected claw-free graph on n vertices with the minimum degree δ, is at most $n + c\ \delta^{4/3}\log^{1/3}\delta$, where c is a constant.

Keywords: clique covering, clique cover number, claw-free graphs, Kneser representation, line graph of hypergraph, set representation.

1 Introduction

Throughout the paper, all graphs are finite and simple and the term *clique* stands for a set of pairwise adjacent vertices as well as the corresponding induced complete subgraph. In addition, by a *biclique* we mean a complete bipartite subgraph. In literature, different variants of edge covering of graphs have been explored, among which, the clique covering and biclique covering are widely studied. A *clique* (*resp. biclique*) *covering* of a graph G is a family $\mathcal{C}$ of cliques (resp. bicliques) of G such that every edge of G belongs to at least one clique (resp. biclique) in $\mathcal{C}$. The *clique* (*resp. biclique*) *cover number* of G, denoted by $\mathrm{cc}(G)$ (resp. $\mathrm{bc}(G)$), is defined as the smallest number of cliques (resp. bicliques) in a clique covering (resp. biclique covering) of G. These concepts turn out to have several relations and applications to a large variety of theoretical and applied problems including set intersection representations of graphs, communication complexity of Boolean functions and encryption key management. For a review on clique and biclique covering see [16, 19].

In contrast to the clique covering problem which is aimed at minimizing the "total" number of cliques comprising a clique covering, in this paper, we are interested in minimizing the maximum number of cliques which are incident with "each vertex". Let us make the notion more accurate. Given a clique covering $\mathcal{C}$ of a graph $G = (V, E)$, for every vertex $x \in V$, the *valency* of x (with respect to $\mathcal{C}$), denoted by $v_{\mathcal{C}}(x)$, is defined to be the number of cliques in $\mathcal{C}$ containing x. The *valency* of the clique covering $\mathcal{C}$ is the maximum valency of all vertices of G with respect to $\mathcal{C}$. The clique covering $\mathcal{C}$ is called a $k-$*clique covering* if its valency is at most k, i.e. every vertex of G belongs to at most k cliques within $\mathcal{C}$. Among all clique coverings of G, we are interested in finding a clique covering of minimum valency. The smallest number k being the valency of a clique covering of G, is called the *local clique cover number* of G and is denoted by $\mathrm{lcc}(G)$. We have,

$$\mathrm{lcc}(G) := \min_{\mathcal{C}} \max_{x\in V} v_{\mathcal{C}}(x),$$

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where the minimum is taken over all clique coverings of G.

The concept of $k-$clique covering was introduced in [21], during the study of the edge intersection graphs of linear hypergraphs (see Section 2). The problem of finding the local clique cover number of a graph appears to have a number of interesting interconnections and interpretations to some other well-known problems. We will discuss these relationships in Section 2.

It should be noted that, in the same line of thought, Dong et al. [9] have proposed the local counterpart of the biclique cover number. The *local biclique cover number* of a graph G, denoted by $\mathrm{lbc}(G)$, is defined as the smallest k for which G admits a $k-$biclique covering, i.e. a biclique covering where each vertex is incident with at most k of the bicliques comprising the covering.

A *claw-free graph* is a graph having no complete bipartite graph $K_{1,3}$ as an induced subgraph. Also, a *quasi-line graph* is a graph where the neighbours of each vertex are a union of two cliques. From the definition it is obvious that the family of quasi-line graphs contains line graphs and is subfamily of the family of claw-free graphs. A natural and recently well-studied question is which properties of line graphs can be extended to quasi-line graphs and then to all claw-free graphs (see e.g. [7,8]). Chen et al. in [6] have proved that the clique cover number of quasi-line graphs on n vertices is at most n, and they have conjectured that the same result can be extended to claw-free graphs.

The main effort of this paper is devoted to investigating the clique covering of claw-free graphs. Our motivations to study the local clique covering of claw-free graphs are twofold. Firstly, we will see in Corollary 1 that the local clique cover number of line graphs is at most two. This raises the natural question how large the local clique cover number of a quasi-line graph and a claw-free graph can be. We will answer this question in Section 4. Secondly, let us define

$$\alpha_l(G) := \max\{t \ : \ K_{1,t} \text{ is an induced subgraph of } G\}.$$

An *independent set* is a subset of mutually non-adjacent vertices. In fact, $\alpha_l(G)$ is the size of the maximum independent set within the neighbourhood of a vertex. Then, in some sense, the parameter $\alpha_l(G)$ can be thought of as the *maximum local independence number* of G. If $K_{1,t}$ is an induced subgraph of G, then $t = \mathrm{lcc}(K_{1,t}) \le \mathrm{lcc}(G)$ and thereby, $\alpha_l(G) \le \mathrm{lcc}(G)$. Though this bound can be tight, e.g. for the cases $\mathrm{lcc}(G) = \Delta(G)$ (see Proposition 8), it can also be very loose. For instance, let $G_t = K_{2,2,\ldots,2}$ be the $t-$partite complete graph. Then $\alpha_l(G_t) = 2$, however, we will see that $\mathrm{lcc}(G_t) > (1/2)\log t$ (see Corollary 4). This gives rise the question that for every fixed t, how large the lcc of a graph G can be, whenever $\alpha_l(G) \le t$. In fact, investigating the lcc of claw-free graphs (as graphs G with $\alpha_l(G) \le 2$) can be perceived as the first step toward answering the above question for general t.

Now, let us give an overview of the organization of forthcoming sections. In Section 2, we review the relationships between the local clique cover number and three well-known areas, namely, line graph of hypergraphs, intersection representation and Kneser representation. In Section 3, we derive some basic bounds for the lcc in terms of the maximum degree and the maximum clique number. In addition, we characterize all graphs G where $\mathrm{lcc}(G) = \Delta(G)$. Section 4 deals with the clique covering of claw-free graphs. In particular, we prove that lcc of a claw-free graph is at most $c\,\Delta/\log\Delta$, where Δ is the maximum degree and c is a constant. Moreover, we prove that this bound is the best possible up to a constant factor. Finally, regarding a conjecture by Chen et al. [6], we prove that the clique cover number of a claw-free graph on n vertices is at most $n + c\,\delta^{4/3}\log^{1/3}\delta$, where δ is the minimum degree and c is a constant.

2 Interactions and interpretations

As we mentioned before, the local clique cover number may be expressed using other graph parameters. In the following, we provide an overview of three important problems associated to the local clique cover number, namely, line graph of hypergraphs, intersection representation and Kneser representation of graphs.

Line graph of hypergraphs. Given a hypergraph $H = (V, \mathcal{F})$, the *line graph* or *edge intersection graph* of H, denoted by $L(H)$, is a simple graph whose vertices correspond to the edges of H and a pair of vertices in $L(H)$ are adjacent if and only if their corresponding edges in H intersect. For an arbitrary graph G, the inverse image $L^{-1}(G)$ is the set of all hypergraphs H where $L(H) = G$.

In [4], the concept of line graph of hypergraphs has been described in terms of clique covering. For this, let $\mathcal{C}$ be a clique covering for a graph G and for each vertex $x \in V(G)$, let $\mathcal{C}_x$ be the set of all cliques in $\mathcal{C}$ which contain x. Define the hypergraph $H_{\mathcal{C}}$ with the vertex set $\mathcal{C}$ and the edge set $\{\mathcal{C}_x \ : \ x \in V(G)\}$. It can be easily seen that a graph G is the line graph of a hypergraph H if and only if there is a clique covering $\mathcal{C}$ for G such that $H \cong H_{\mathcal{C}}$. From this, one may deduce that every simple graph is the line graph of a hypergraph.

A hypergraph H is called $k-$*uniform* if its edges have the same cardinality k. The class of line graphs of $k-$uniform hypergraphs is denoted by L_k. Note that, in every $k-$clique covering, one may make the valency of all vertices equal to k by adding some dummy single-vertex cliques. Thus, one may see that a graph G belongs to the class L_k if and only if it admits a $k-$clique covering (see [21], for more details). Hence, the following equality holds,

$$\mathrm{lcc}(G) = \min\{k \ : \ G \in L_k\}. \tag{1}$$

Consequently, the problem "Is $\mathrm{lcc}(G) \leq k$?" reduces to "Is $G \in L_k$?", i.e. "Does there exist a $k-$uniform hypergraph whose line graph is isomorphic to G?". It is clear that $\mathrm{lcc}(G) = 1$ if and only if G is a disjoint union of cliques. Also, by (1), we have the following corollary.

Corollary 1. *For every graph G, $\mathrm{lcc}(G) \leq 2$ if and only if G is the line graph of a multigraph.*

The class L_2 turns out to have a characterization by a list of 7 forbidden induced subgraphs and a polynomial time algorithm has been found to decide if $G \in L_2$ (see [5, 13]). In contrast to the case $k = 2$, the situation for the case $k \geq 3$ is completely different. Lovász in [14] has proved that there is no characterization for the class L_3 by a finite list of forbidden induced subgraphs. Also, it has been proved that the decision problems "Is $G \in L_k$?" for fixed $k \geq 4$ and the problem of recognizing line graphs of $3-$uniform hypergraphs without multiple edges are NP-complete [18]. This leads us to the following hardness results for lcc.

Corollary 2. (i) *The decision problem "Is $\mathrm{lcc}(G) \leq 2$?" is polynomially solvable.*

(ii) *For every fixed $k \geq 4$, the decision problem "Is $\mathrm{lcc}(G) \leq k$?" is an NP-complete problem.*

(iii) *The decision problem "Does there exist a $3-$clique covering for G, such that no two distinct vertices appear in exactly the same set of cliques?", is an NP-complete problem.*

Also note that the complexity status of the decision problem "Is $\mathrm{lcc}(G) \leq 3$?", in general, remains an open question.

Intersection representation. An *intersection representation* for graph $G = (V, E)$ is a representation of each vertex by a set, such that every two distinct vertices are adjacent if and only if their corresponding sets intersect. In other words, it is a function $\mathfrak{R} : V \to \mathcal{P}(L)$, where L is a set of labels, such that for every two distinct vertices $x, y \in V$,

$$x \sim y \ \text{ if and only if } \ \mathfrak{R}(x) \cap \mathfrak{R}(y) \neq \emptyset.$$

For each $i \in L$, the vertices being represented by the sets containing i form a clique in G. On the other hand, every clique covering $\mathcal{C}$ induces an intersection representation which assigns to each vertex x the set $\mathcal{C}_x$ (see above). This sets up a one-to-one correspondence between the clique coverings of G and the intersection representations for G (see e.g. [15], for more details). A $k-$*set representation* is an intersection representation $\mathfrak{R}$ such that for each $x \in V$, $\mathfrak{R}(x)$ is of size at most k. Therefore, one may conclude that $\mathrm{lcc}(G)$ is the minimum k for which G admits a $k-$set representation. Indeed,

$$\mathrm{lcc}(G) = \min_{\mathfrak{R}} \max_{x \in V} |\mathfrak{R}(x)|. \tag{2}$$

Kneser representation. Given positive integers n and k, $n \geq 2k$, the Kneser graph with paremeters n, k, denoted by $KG(n,k)$, is the graph with the vertex set $[n]^k$, the set of all $k-$subsets of $[n] := \{1,\ldots,n\}$, such that a pair of vertices are adjacent if and only if the corresponding subsets are disjoint. It can be seen that every graph is an induced subgraph of a Kneser graph [11]. Hamburger et al. have proposed the question that what is the smallest k for which G is the induced subgraph of a Kneser graph $KG(n,k)$, for some integer n. The minimum k for which there exists some integer n such that G is the induced subgraph of $KG(n,k)$ is called the *Kneser index* of G and is denoted by $\iota^K(G)$ [11]. The Kneser index of cycles, paths and hypercubes are studied in [11]. Proposition 3 states a relation between the lcc of a graph G and the Kneser index of its complement $\overline{G}$.

A pair of adjacent vertices $x, y \in V$, are called *twins*, if $N(x) \setminus \{y\} = N(y) \setminus \{x\}$, where $N(x)$ stands for the set of neighbours of x. A graph G is called *twin-free* if it has no twins.

Proposition 3. *For every graph G, we have $\operatorname{lcc}(G) \leq \iota^K(\overline{G}) \leq \operatorname{lcc}(G)+1$. Furthermore, $\operatorname{lcc}(G) = \iota^K(\overline{G})$ whenever G is a twin-free graph.*

Proof. First we prove the fact that an injective $k-$set representation for G exists if and only if $\iota^K(\overline{G}) \leq k$. It is obvious that $\iota^K(\overline{G}) \leq k$ implies that G has an injective $k-$set representation. Conversely, if G admits an injective $k-$set representation $\mathfrak{R}$, then one can add dummy new labels to the sets $\mathfrak{R}(x)$, $x \in V$, in order to make all of them of the same size k. This shows that each vertex can be represented by a distinct $k-$set, where only the sets corresponding to adjacent vertices intersect. Hence, $\overline{G}$ is an induced subgraph of $KG(n,k)$.

Furthermore, given a $k-$set representation of G, one may find an injective $(k+1)-$set representation by adding different new labels to the sets (one label to each set). Hence, by the above fact, $\operatorname{lcc}(G) \leq \iota^K(\overline{G}) \leq \operatorname{lcc}(G)+1$.

Finally, note that every intersection representation of a twin-free graph is indeed injective, because distinct vertices should be represented by distinct sets of labels. Therefore, for twin-free graph G, we have $\operatorname{lcc}(G) = \iota^K(\overline{G})$. □

In [11] it is shown that if a graph G contains an induced matching of size t, and $t > \binom{2k}{k}$, for some k, then $\iota^K(G) > k$. Thus, by Proposition 3, we have the following corollary which shows that there exist graphs with large lcc and small α_l.

Corollary 4. *If $G_t = K_{2,2,\ldots,2}$ is the $t-$partite complete graph, then we have $\operatorname{lcc}(G_t) > (1/2)\log t$.*

Remark 5. Every graph G has a twin-free induced subgraph H for which $\operatorname{lcc}(G) = \operatorname{lcc}(H)$. To see this, note that the relation of being twins is an equivalence relation on $V(G)$. Let H be the induced subgraph of G obtained by deleting all but one vertex from each of the equivalence classes. It is evident that $\operatorname{lcc}(H) \leq \operatorname{lcc}(G)$. On the other hand, every $k-$clique covering for H can be extended to a $k-$clique covering for G substituting every vertex by its corresponding equivalence class. Hence, $\operatorname{lcc}(G) = \operatorname{lcc}(H)$, as desired. ▲

3 Basic bounds

In this section, we provide simple lower and upper bounds for $\operatorname{lcc}(G)$ in terms of the maximum degree and the maximum clique number. In addition, we characterize the case when the upper bound achieved.

Proposition 6. *For every graph G with maximum degree Δ and maximum clique number ω, we have*

$$\frac{\Delta}{\omega-1} \leq \operatorname{lcc}(G) \leq \Delta. \tag{3}$$

Proof. The upper bound is a straightforward consequence of the fact that all edges of G comprise a Δ-clique covering. For the lower bound, let $k = \operatorname{lcc}(G)$ and $\mathcal{C}$ be a $k-$clique covering. Fix a vertex

x and define $\mathcal{C}_x := \{C \in \mathcal{C} \ : \ x \in C\}$. By the definition of lcc, we have $|\mathcal{C}_x| \le k$. Furthermore, each edge incident with x is contained in some clique $C \in \mathcal{C}_x$. Therefore,

$$\deg(x) \le \sum_{C \in \mathcal{C}_x} |C \setminus \{x\}| \le |\mathcal{C}_x|(\omega - 1) \le k(\omega - 1) = \mathrm{lcc}(G)(\omega - 1).$$

As x was arbitrary, the desired bound follows. □

Using the above proposition, we may determine the exact value of $\mathrm{lcc}(G)$ for triangle-free graphs.

Corollary 7. *For every triangle free graph G with maximum degree Δ, we have $\mathrm{lcc}(G) = \Delta$.*

The following proposition characterize all the graphs for which $\mathrm{lcc}(G) = \Delta$.

Proposition 8. *Given a graph G with maximum degree Δ, we have $\mathrm{lcc}(G) = \Delta$ if and only if there exists a vertex $x \in V(G)$ of degree Δ, such that $N(x)$ is an independent set, that is $\alpha_l(G) = \Delta$.*

Proof. Suppose that there exists a vertex $x \in V(G)$ of degree Δ, such that $N(x)$ is an independent set. Therefore, in every clique covering of G, we need Δ cliques to cover the edges incident with x. Thus, $\mathrm{lcc}(G) \ge \Delta$.

Conversely, assume that $\mathrm{lcc}(G) = \Delta$. Let $\mathcal{C}$ be a $\Delta-$covering for which $\sum_{C \in \mathcal{C}} |C|$ is minimum. Also, let x be a vertex which is contained in Δ cliques of $\mathcal{C}$. Observe that, $\deg(x) = \Delta$. Otherwise, the edges incident with x can be covered by at most $\Delta - 1$ cliques in $\mathcal{C}$. By excluding x from the extra cliques, we obtain a new $\Delta-$covering, contradicting the minimality assumption. The same argument shows that each edge incident with x is actually a clique in $\mathcal{C}$. Now, it is enough to prove that $N(x)$ is an independent set. Assume, to the contrary, that $y, z \in N(x)$ are adjacent. In this case, one may replace the cliques $\{x, y\}, \{x, z\} \in \mathcal{C}$ by the clique $\{x, y, z\}$ to obtain a new $\Delta-$covering, contradicting the minimality assumption. Hence, the assertion holds. □

4 LCC of claw-free graphs

In this section, we focus on the class of claw-free graphs and particularly we concentrate on the question of how large the lcc of a claw-free graph can be. Although the upper bound $\mathrm{lcc}(G) \le \Delta$ in Proposition 6 is tight, we show that this bound can be asymptotically improved for claw-free graphs. In this regard, for every integer k, let us define

$$f(k) := \max\{\mathrm{lcc}(G) \ : \ G \text{ is claw-free and } \Delta(G) \le k\}.$$

Now, the question is that how the function $f(k)$ behaves. The same question could be also asked for the quasi-line graphs. Let us define

$$g(k) := \max\{\mathrm{lcc}(G) \ : \ G \text{ is a quasi-line graph and } \Delta(G) \le k\}.$$

In the following, we determine asymptotic behaviors of the functions $f(k)$ and $g(k)$, by proving that for some constants c_1, c_2,

$$c_1 \, \frac{k}{\log k} \le g(k) \le f(k) \le c_2 \, \frac{k}{\log k}. \tag{4}$$

This section is devoted to establish (4) (here, we make no attempt to find the best possible constants). In order to prove the lower bound, for every integer k, one ought to provide a quasi-line graph G where $\Delta(G) \le k$ and $\mathrm{lcc}(G) \ge c_1 \, k/\log k$. This is exactly what we are going to do in the following theorem. A graph is called *cobipartite* if its complement is bipartite. It is evident that every cobipartite graph is a quasi-line graph.

Theorem 9. *For every integer n, there exists a cobipartite graph G on n vertices such that*

$$\mathrm{lcc}(G) > \frac{1}{4}(1 - o(1))\frac{n}{\log n},$$

where $o(1)$ tends to 0, as n goes to infinity.

Proof. For fixed integers $n, t, t \leq n^2/8$, let G be a graph on n vertices whose complement is a bipartite $n/2 \times n/2$ graph (i.e. $V(G)$ is the disjoint union of two cliques of size $n/2$). Also, let $\mathfrak{R} : V(G) \to \mathcal{P}(L)$ be a t-set representation for G with the label set L (see Section 2 for the definition). Then, without loss of generality, we can assume that $|L| \leq n^2/4$, because for each edge between two parts, we need at most one new label in L.

Now, on the one hand, the number of all t-set representations with $n^2/4$ labels for a graph on n vertices, is at most

$$\left[\sum_{i=0}^{t} \binom{n^2/4}{i}\right]^n \leq t^n \left(\frac{en^2}{4t}\right)^{nt},$$

and, on the other hand, the number of all bipartite $n/2 \times n/2$ graphs (with vertices in one part labelled by $1, 2, \ldots, n/2$ and vertices in the other part labelled by $n/2 + 1, \ldots, n$) is $2^{n^2/4}$. We set t such that for sufficiently large n,

$$2^{n^2/4} > t^n \left(\frac{en^2}{4t}\right)^{nt}. \tag{5}$$

This ensures the existence of a cobipartite graph G which admits no t-set representation and consequently $\mathrm{lcc}(G) > t$. It only remains to do some tedious computations to check Inequality (5), for $t := 1/4(1 - o(1))\ n/\log n$. □

Proving the upper bound in (4) is more difficult. In the following, we prove that for every claw-free graph G with maximum degree Δ, $\mathrm{lcc}(G) \leq c\ \Delta/\log \Delta$, where c is a constant. Towards achieving this objective, we apply a result of Erdős et al. about the decomposition of a graph into complete bipartite graphs [10], along with a well-known result of Ajtai et al. on the independence number of triangle-free graphs. Let us recall these in the following two theorems.

Theorem A. [10] *The edge set of every graph on n vertices can be partitioned into complete bipartite subgraphs* (*bicliques*) *such that each vertex is contained in at most* $c_0\ n/\log n$ *of the bicliques, i.e.* $\mathrm{lbc}(G) \leq c_0\ n/\log n$, *where* c_0 *is a constant.*

Theorem B. [1, 2, 20] *Every triangle-free graph on n vertices contains an independent set of size* $\sqrt{2}/2(1 - o(1))\ \sqrt{n \log n}$.

Since the property of being triangle-free is hereditary with respect to subgraphs, one can iteratively apply Theorem B and omit the large independent sets from the vertex set, thereby obtaining a proper coloring for the graph (see e.g. [12, 17]). Therefore, the following theorem is a consequence of Theorem B.

Theorem C. *The vertex set of every triangle-free graph on n vertices can be properly colored by at most* $2\sqrt{2}(1 + o(1))\sqrt{n/\log n}$ *colors, such that each color class is of size at most* $\sqrt{2}/2\sqrt{n \log n}$.

Now, with all these results in hand, we are ready to prove the main theorem of this section.

Theorem 10. *For every claw-free graph G with maximum degree* Δ, *we have* $\mathrm{lcc}(G) \leq c\ \Delta/\log \Delta$, *where c is a constant.*

Proof. Fix a claw-free graph $G = (V, E)$ with maximum degree Δ. We provide a clique covering for G, where each vertex is contained in at most $c\ \Delta/\log \Delta$ cliques.

Let $I \subset V$ be a maximal independent set of vertices in G and fix a vertex $u \in I$. Since G is claw-free, the subgraph of G induced by the neighbourhood of u, $G[N(u)]$, has no independent set of size three. Thus, its complement, $\overline{G[N(u)]}$, is a triangle-free graph on at most Δ vertices. Then, by Theorem C, the vertex set of $\overline{G[N(u)]}$ can be properly colored by at most $2\sqrt{2}(1 + o(1))\sqrt{\Delta/\log \Delta}$ colors, such that each color class is of size at most $\sqrt{2}/2\sqrt{\Delta \log \Delta}$. Let $C_1, \ldots, C_\ell$ be all color classes which are obtained in the such coloring of $\overline{G[N(u)]}$ (see Figure 1). For each i, $1 \leq i \leq \ell$, the set $C_i \cup \{u\}$ forms a clique in G. Therefore, all edges incident with u can be covered by at most $2\sqrt{2}(1 + o(1))\sqrt{\Delta/\log \Delta}$ cliques.

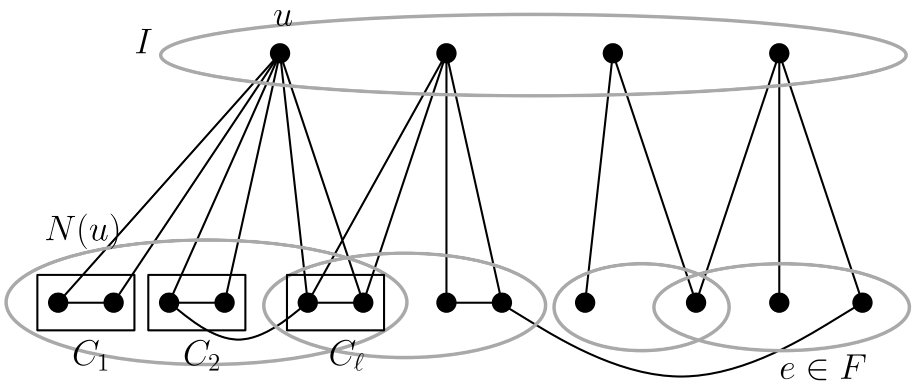

Figure 1: A schema of the claw-free graph G.

On the other hand, by virtue of Theorem A, for every i, j, $1 \leq i < j \leq \ell$, the edges of G which lie between the color classes C_i, C_j can be partitioned into bicliques such that each vertex in $C_i \cup C_j$ is contained in at most $2\sqrt{2}c_0\sqrt{\Delta \log \Delta}/\log \Delta$ of the bicliques. Since each C_i induces a clique in G, the vertex set of each of these bicliques induces a clique in G. Hence, all these cliques for every $1 \leq i < j \leq \ell$, cover all the edges in $G[N[u]]$, where $N[u] = N(u) \cup \{u\}$ is the closed neighbourhood of u. Let us denote this clique covering by $\mathcal{C}_u$. In the clique covering $\mathcal{C}_u$, each vertex $v \in N[u]$ is contained in at most

$$2\sqrt{2}c_0 \frac{\sqrt{\Delta \log \Delta}}{\log \Delta}\; 2\sqrt{2}(1 + o(1))\sqrt{\frac{\Delta}{\log \Delta}} = 8c_0(1 + o(1))\frac{\Delta}{\log \Delta}$$

of the cliques. For each $u \in I$, let us cover the edges in $G[N[u]]$ by the clique covering $\mathcal{C}_u$ and define $\mathcal{C} := \cup_{u \in I}\mathcal{C}_u$. Since G is claw-free and I is a maximal independent set, each vertex $v \in V$ has 1 or 2 neighbours in I. Therefore, every vertex $v \in V$ is contained in at most $16c_0(1 + o(1))\Delta/\log \Delta$ of the cliques comprising $\mathcal{C}$.

The cliques in $\mathcal{C}$ cover all the edges in $G[N[u]]$, for all $u \in I$, but it does not necessarily cover all the edges of G. Now, let $F \subset E$ be the set of all the edges which are not covered by the cliques of $\mathcal{C}$ and let H be the subgraph of G induced by F. For the remaining of the proof, we look for a suitable clique covering of H and count the contribution of each vertex in this covering.

In order to provide a desired clique covering for H, we have to describe the structure of the subgraph H. For this purpose, first, we establish a sequence of facts regarding H.

Since all the edges covered by the cliques in $\mathcal{C}$ are exactly the ones in $G[N[u]]$, for all $u \in I$, we have the following fact.

Fact 1. For every edge $e = xy \in E$, we have $e = xy \in F$ if and only if the vertices x, y have no common neighbour in I.

Assume $x \in V(H)$ and y_1, y_2 are two neighbours of x in H, i.e. $y_1, y_2 \in N_H(x)$. The vertex x has also a neighbour u in I. By Fact 1, y_1, y_2 are not adjacent with u. Hence, due to claw-freeness of G, y_1, y_2 must be adjacent in G. Consequently, the following fact holds.

Fact 2. For every vertex $x \in H$, its neighbours in H, $N_H(x)$ induces a clique in G.

With the same argument (using Fact 1 along with the claw-freeness of G), one may prove the following fact.

Fact 3. Every non-isolated vertex of H has exactly one neighbour in the set I.

Assuming $I = \{u_1, \ldots, u_\alpha\}$, with the aid of Facts 1 and 3, the non-isolated vertices of H can be partitioned into α disjoint sets $N_1, \ldots, N_\alpha$, where

$$N_i := \{x \in V(H) \;:\; x \text{ is non-isolated and is adjacent to } u_i\}, \quad 1 \leq i \leq \alpha.$$

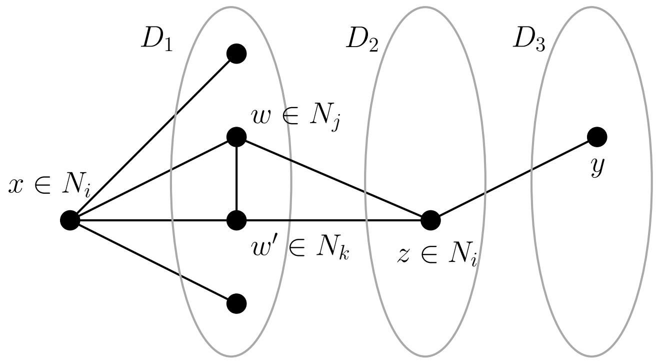

Figure 2: A schema of a connected component of H, assuming, to the contrary, that $D_3 \neq \emptyset$.

Now suppose that $x \in V(H)$ and $y, z \in N_H(x)$, where $y \in N_i$ and $z \in N_j$, for some $i \neq j$. By Fact 2, we know that y, z are adjacent in G. But y, z has no common neighbour in I. Thus, due to Fact 1, $yz \in F$. Hence, the following assertion holds.

Fact 4. If $x \in V(H)$ and $y, z \in N_H(x)$, where $y \in N_i$ and $z \in N_j$, for some $i \neq j$, then yz is an edge in F.

Now, we are ready to prove the following claim concerning the structure of the graph H.

Claim. Every connected component of H is either

- an isolated vertex, or
- a bipartite graph with bipartition (N_i, N_j), for some $1 \leq i < j \leq \alpha$, or
- a graph on at most 2Δ vertices whose diameter is at most two.

Proof of Claim. Fix a vertex $x \in N_i$ and for $d \in \mathbb{Z}^+ \cup \{0\}$, let $D_d := \{y \in V(H) \ : \ d_H(x, y) = d\}$. First, we prove that if $D_d \subseteq N_j$, for some d, j, then $D_{d+2} \subseteq N_j$. To see this, assuming $D_d \subseteq N_j$, let $y \in D_{d+2}$. Then, y has a neighbour y' in D_{d+1} and y' has a neighbour y'' in D_d, where $y'' \in N_j$. If $y \notin N_j$, then, due to Fact 4, y, y'' should be adjacent in H, which is a contradiction. Thus, $y \in N_j$.

Hence, since $D_0 = \{x\} \subseteq N_i$, we have $D_{2d} \subseteq N_i$, for every d. If, in addition, $D_1 \subseteq N_j$, for some j, then $D_{2d+1} \subseteq N_j$, for all d. This shows that, in case $D_1 \subseteq N_j$, the connected component of H containing x, is a bipartite graph with bipartition (N_i, N_j).

Now assume that $D_1 \nsubseteq N_j$, for all j. With this assumption, we prove that the set D_3 is empty. Assume, to the contrary, that $y \in D_3$ and let z be a neighbour of y in D_2 and also let w be a neighbour of z in D_1 (See Figure 2). We have $w \notin N_i$ (because of Fact 1). Assume $w \in N_j$, for some $j \neq i$. Since $D_1 \nsubseteq N_j$, there is a vertex $w' \in D_1 \cap N_k$, for some $k \neq j$. Due to Fact 4, w and w' are adjacent in H. Moreover, $z \in N_i$, again by Fact 4, z and w' are adjacent in H. Now, $y \in D_3$ are not adjacent to $w, w' \in D_1$ in H. Hence, by Fact 4, $y \in N_j \cap N_k$. This contradicts with the fact that N_j and N_k are disjoint. Consequently, the set D_3 is empty and the connected component of H containing x has diameter at most two. On the other hand, $|D_1| = |N_H(x)| \leq \Delta$ and $|D_0 \cup D_2| \leq |N_i| \leq \deg(u_i) \leq \Delta$. Hence, the connected component has at most 2Δ vertices. ■

Now, we get back into the proof of Theorem 10. Consider a nontrivial connected component of the graph H. By the above claim, either it is a graph on at most 2Δ vertices whose diameter is at most two, or it is a bipartite graph with bipartition (N_i, N_j), for some $1 \leq i < j \leq \alpha$. Since $|N_i| \leq \deg(u_i) \leq \Delta$, the latter case has also at most 2Δ vertices. Hence, every connected component of H is of size at most 2Δ. Therefore, by Theorem A, one may construct a biclique covering for every connected component of H where each of its vertices belongs to at most $2c_0\Delta/\log\Delta$ bicliques. Because of Fact 2, every biclique of H induces a clique in G. As a consequence, one may provide a collection of cliques of G which cover all the edges of H and each vertex belongs to at most $2c_0\Delta/\log\Delta$ of these cliques. This collection together with the clique covering $\mathcal{C}$ provides a clique covering for G, for which every vertex contributes in at most $18c_0(1+o(1))\Delta/\log\Delta$ number of cliques, thereby establishing Theorem 10. □

5 CC of claw-free graphs

Chen et al. in [6] proved that the clique cover number of quasi-line graphs on n vertices is at most n, and they conjectured that the same result can be extended to claw-free graphs. Toward this conjecture, in Theorem 12, we provide an upper bound for the clique cover number of claw-free graphs. For this purpose, we need the following theorem from Alon.

Theorem D. [3] *For every graph G on n vertices, we have $\mathrm{cc}(G) \leq c\,\Delta(\overline{G})^2 \log n$, where c is a constant.*

Note that the complement of every triangle-free graph is claw-free. In the next theorem we prove an upper bound for the clique cover number of the complement of triangle-free graphs.

Theorem 11. *If G is a triangle-free graph on n vertices, then $\mathrm{cc}(\overline{G}) \leq c\,n^{4/3} \log^{1/3} n$, where c is a constant.*

Proof. We prove the theorem by induction on n. If $\Delta(G) \leq n^{2/3}/\log^{1/3} n$, then, by Theorem D, $\mathrm{cc}(\overline{G}) \leq c\,n^{4/3} \log^{1/3} n$ and we are done. Thus, assume that $\Delta(G) \geq n^{2/3}/\log^{1/3} n$. Let x be a vertex of degree d in G, where $d \geq n^{2/3}/\log^{1/3} n$ and $N_G(x)$ be the set of neighbours of x in G. Since G is triangle-free, $N_G(x)$ is a clique in $\overline{G}$. For every vertex $y \in N_{\overline{G}}(x)$, Define

$$C_y := \{y\} \cup (N_{\overline{G}}(y) \cap N_G(x)).$$

Each C_y is a clique in $\overline{G}$ and the collection of cliques $\{C_y \ : \ y \in N_{\overline{G}}(x)\}$ along with the clique $N_G(x)$ cover all edges of $\overline{G}$ which have at least one end in $N_G(x)$. Now, let G' be the subgraph of G induced by $V(G) \setminus N_G(x)$. By the induction hypothesis, we have,

$$\begin{aligned}
\mathrm{cc}(\overline{G}) &\leq n - d + \mathrm{cc}(\overline{G'}) \leq n - d + c\,(n-d)^{\frac{4}{3}} \log^{\frac{1}{3}}(n-d) \\
&\leq n - \frac{n^{\frac{2}{3}}}{\log^{\frac{1}{3}} n} + c\left(n - \frac{n^{\frac{2}{3}}}{\log^{\frac{1}{3}} n}\right)^{\frac{4}{3}} \log^{\frac{1}{3}} n \\
&\leq n - \frac{n^{\frac{2}{3}}}{\log^{\frac{1}{3}} n} + c\left(1 - \frac{1}{(n\log n)^{\frac{1}{3}}}\right) n^{\frac{4}{3}} \log^{\frac{1}{3}} n \\
&\leq c\,n^{\frac{4}{3}} \log^{\frac{1}{3}} n.
\end{aligned}$$

□

We close the paper, by proving the following upper bound for the clique cover number of claw-free graphs.

Theorem 12. *If G is a connected claw-free graph on n vertices with the minimum degree δ, then $\mathrm{cc}(G) \leq n + c\,\delta^{4/3} \log^{1/3} \delta$, where c is a constant.*

Proof. Let G be a connected claw-free graph on n vertices and x be a vertex of the minimum degree δ. Also, for every $i \geq 0$, let V_i be the set of vertices of distance i from x; for instance $V_0 = \{x\}$, $V_1 = N_G(x)$. For every vertex $y \neq x$ and $i \geq 0$, where $y \in V_i$, let $N_-(y)$, $N_0(y)$ and $N_+(y)$ be the sets of neighbours of y in V_{i-1}, V_i and V_{i+1}, respectively. Now for every $y \neq x$, define

$$C_y := \{y\} \cup \{z \in N_0(y) \cup N_+(y) \ : \ N_-(z) \cap N_-(y) = \emptyset\}.$$

It can be seen that C_y is a clique in G, because there is a vertex $w \in N_-(y)$ which is not adjacent to the vertices of C_y and G is claw-free. Moreover, the collection $\{C_y \ : \ y \neq x\}$ covers all edges of G except the ones whose both ends lie in $V_0 \cup V_1$.

Let $G' = G[V_1]$ be the induced graph on V_1. Since G is claw-free, $\overline{G'}$ is a triangle-free graph on δ vertices. Therefore, by Theorem 11, all edges of G' can be covered by at most $c\,\delta^{4/3} \log^{1/3} \delta$ cliques, where c is a constant. By adding x to these cliques, we can find a clique covering for $G[V_0 \cup V_1]$. This clique covering along with the cliques C_y, $y \neq x$, form a clique covering for G with at most $n - 1 + c\,\delta^{4/3} \log^{1/3} \delta$ cliques. □

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980.

[2] M. Ajtai, J. Komlós, and E. Szemerédi. A dense infinite Sidon sequence. *European J. Combin.*, 2(1):1–11, 1981.

[3] N. Alon. Covering graphs by the minimum number of equivalence relations. *Combinatorica*, 6(3):201–206, 1986.

[4] C. Berge. *Hypergraphs*, volume 45 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1989. Combinatorics of finite sets, Translated from the French.

[5] J.-C. Bermond and J. C. Meyer. Graphe représentatif des arêtes d'un multigraphe. *J. Math. Pures Appl. (9)*, 52:299–308, 1973.

[6] Guantao Chen, Michael S. Jacobson, André E. Kézdy, Jenö Lehel, Edward R. Scheinerman, and Chi Wang. Clique covering the edges of a locally cobipartite graph. *Discrete Math.*, 219(1-3):17–26, 2000.

[7] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 153–171. Cambridge Univ. Press, Cambridge, 2005.

[8] M. Chudnovsky and P. Seymour. Claw-free graphs VI. colouring. *J. Comb. Theory Ser. B*, 100(6):560–572, November 2010.

[9] J. Dong and Y. Liu. On the decomposition of graphs into complete bipartite graphs. *Graphs Combin.*, 23:255–262, 2007.

[10] P. Erdős and L. Pyber. Covering a graph by complete bipartite graphs. *Discrete Math.*, 170(1-3):249–251, 1997.

[11] P. Hamburger, A. Por, and M. Walsh. Kneser representations of graphs. *SIAM J. Discrete Math.*, 23(2):1071–1081, 2009.

[12] J. H. Kim. The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms*, 7(3):173–207, 1995.

[13] Philippe G. H. Lehot. An optimal algorithm to detect a line graph and output its root graph. *J. Assoc. Comput. Mach.*, 21:569–575, 1974.

[14] L. Lovász. *Problem 9: Beiträge zur Graphentheorie und deren Anwendungen.* Technische Hochschule Ilmenau, 1977. International Colloquium held in Oberhof.

[15] T. A. McKee and F. R. McMorris. *Topics in intersection graph theory.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[16] Sylvia D. Monson, Norman J. Pullman, and Rolf Rees. A survey of clique and biclique coverings and factorizations of $(0,1)$-matrices. *Bull. Inst. Combin. Appl.*, 14:17–86, 1995.

[17] A. Nilli. Triangle-free graphs with large chromatic numbers. *Discrete Math.*, 211(1-3):261–262, 2000.

[18] S. Poljak, V. Rödl, and D. Turzík. Complexity of representation of graphs by set systems. *Discrete Appl. Math.*, 3(4):301–312, 1981.

[19] N. J. Pullman. Clique coverings of graphs—a survey. In *Combinatorial mathematics, X (Adelaide, 1982)*, volume 1036 of *Lecture Notes in Math.*, pages 72–85. Springer, Berlin, 1983.

[20] J. B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46(1):83–87, 1983.

[21] P.V. Skums, S.V. Suzdal, and R.I. Tyshkevich. Edge intersection graphs of linear 3-uniform hypergraphs. *Discrete Math.*, 309(11):3500–3517, 2009.