Critical Hamiltonian Connected Graphs

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Abstract

A graph G is Hamiltonian connected, if there is a Hamiltonian path between every two distinct vertices of G. A Hamiltonian connected graph G is called critical Hamiltonian connected (CHC), if for every edge e in G, graph $G-e$ is not Hamiltonian connected. In this paper, we study the properties of CHC graphs.

Keywords: Hamiltonian connected, Critical Hamiltonian connected, Harary graph.

1 Introduction

Through the paper all graphs are finite, simple and connected. The notations δ and Δ are used for minimum and maximum degree of graph, respectively, and $d(v)$ denotes the degree of vertex v in G. We write $u \leftrightarrow v$ ($u \not\leftrightarrow v$) when u and v are adjacent (not adjacent). The set of adjacent vertices to vertex v is denoted by $N(v)$. A spanning cycle and a spanning path in a graph are called Hamiltonian cycle and Hamiltonian path. A Hamiltonian graph is a graph with a Hamiltonian cycle. A graph is called Hamiltonian connected if there is a Hamiltonian path between every two vertices of it (see [4, 5, 6]). For the definitions and notations, we follow [7].

The number of edges in a Hamiltonian connected graph is quite large. A formula for minimum number of edges needed to guarantee a graph to be Hamiltonian connected is found in terms of the order and minimum degree of graph in [2]. In [6], it is proved that if the size of graph G of order n is at least $\binom{n-1}{2}+2$, then G is Hamiltonian and if the size of G is at least $\binom{n-1}{2}+3$, then G is Hamiltonian connected. Moon in [5], proved that every vertex in a Hamiltonian connected graph of order $n \geq 4$ has degree at least 3. Hence, every Hamiltonian connected graph of order $n \geq 4$ has at least $\lceil 3n/2 \rceil$ edges. For every positive integer $n \geq 4$, we have a Hamiltonian connected graph of order n and size $\lceil 3n/2 \rceil$ (see Figure 1). For $n = 2k$, $k \geq 3$, graph G_n consists of two cycles $u_1, \ldots, u_k, u_1$ and $v_1, \ldots, v_k, v_1$ and edges u_iv_i, $1 \leq i \leq k$. This graph is called k-prism. For $n = 2k + 1$, $k \geq 3$, graph G_n consists of two cycles $u_1, \ldots, u_k, u_1$ and $v_1, \ldots, v_k, v_{k+1}, v_1$ and edges u_iv_i, $1 \leq i \leq k$, and u_kv_{k+1}. We call this graph $(k, k+1)$-prism.

Fault tolerant Hamiltonian connectivity is another important parameter for graphs as indicated in [3]. A Hamiltonian connected graph G is k edge-fault tolerant Hamiltonian connected if $G - F$ remains Hamiltonian connected for any $F \subseteq E(G)$, with $|F| \leq k$. The edge-fault tolerant Hamiltonian connectivity of a Hamiltonian connected graph G, denoted by $\mathcal{HC}_e(G)$, is the maximum integer k such that G is k edge-fault tolerant Hamiltonian connected.

In this paper, we introduce the concept of critical Hamiltonian connected graph (CHC) which are Hamiltonian connected graphs that are not Hamiltonian connected after discarding any edge of them. Observe that for CHC graph G, $\mathcal{HC}_e(G) = 0$. Here, some necessary conditions for a graph to be CHC are obtained. Also, infinite family of CHC graphs with some given maximum degree are constructed . Finally, as a well known class of graphs, we consider Harary graph $H_{k,n}$ and prove that for $k \geq 4$ and $n \geq 5$, $H_{k,n}$ is Hamiltonian connected and it is not CHC, while for some integer n, $H_{3,n}$ is a CHC graph.

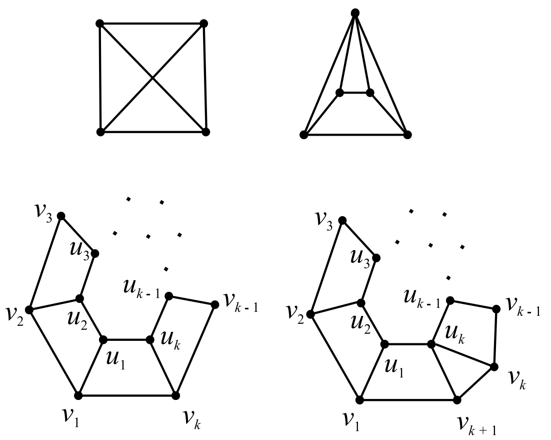

Figure 1: Graph G_n of order $n \geq 4$ and size $\lceil 3n/2 \rceil$.

2 Critical Hamiltonian connected graphs

In the following definition, we consider the Hamiltonian connected graphs in critical state.

Definition 1. *Let G be a Hamiltonian connected graph. An edge e is* **critical Hamiltonian edge** *in G, if for some vertices u and v in $V(G)$ there is no Hamiltonian path from u to v in $G - e$. In this case, we say e is a* **$(\mathbf{u}, \mathbf{v})$-critical Hamiltonian edge**. *A Hamiltonian connected graph G is said* **critical Hamiltonian connected (CHC)***, if every $e \in E(G)$ is critical Hamiltonian edge.*

Proposition 1. *Every Hamiltonian connected graph of order $n \geq 4$, is 3-connected.*

Proof. Let G be a connected graph. If v is a cut vertex of G and B_1 is a block contains v, then there is no Hamiltonian path with end vertices v and a vertex $v_1 \in V(B_1)$. Hence, G is a 2-connected graph.

Now let $S = \{u, v\}$ be a vertex cut of G. Every path with end vertices u and v contains the vertices of at most one of the components of $G - S$. Therefore, there is no Hamiltonian path from u to v. Hence, every vertex cut in G has at least three vertices. ■

Since in every Hamiltonian connected graph of order $n \geq 4$, we have $\delta \geq 3$, every incident edge to a vertex of degree 3 is critical Hamiltonian edge. For example in k-prism, $(k, k+1)$-prism and wheels every edge is critical Hamiltonian edge. Thus, these graphs are CHC. The converse of this fact is not true. In Figure 2, we have a CHC graph with a critical Hamiltonian edge v_5v_{10}, in which v_5 and v_{10} are of degree 4. In fact, edge v_5v_{10} is (v_3, v_9)-critical Hamiltonian edge and also (v_4, v_8)-critical Hamiltonian edge. One of the Hamiltonian paths from v_3 to v_9 is shown by bold edges in Figure 2.

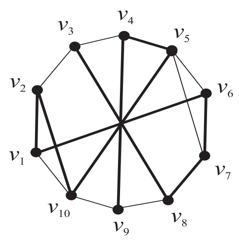

Figure 2: A CHC graph contains an edge with end vertices of degree 4.

Every Hamiltonian connected graph contains a spanning critical Hamiltonian connected subgraph, since we can continue discarding edges by preserving

Hamiltonian connectivity property. The spanning critical Hamiltonian connected subgraph of a graph is not necessarily unique. For example in graph G in Figure 3, $G - e_1$ and $G - e_2 - e_3$ are two critical Hamiltonian connected subgraphs of G. Note that every CHC graph has minimal size, but it may not be minimum.

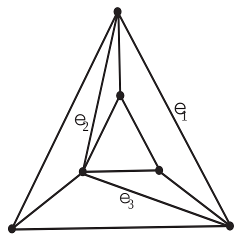

Figure 3: A non-critical Hamiltonian connected graph G.

Observation 1. *If H is a spanning Hamiltonian connected subgraph of a graph G and H is not CHC, then G is not CHC, too.*

A graph G of order $n \leq 4$ is a CHC graph if and only if $G = K_n$. Hence, if u and v are two adjacent vertices in G, then $d(u) + d(v) \leq n + 2$. In the following theorem, this result is obtained for $n \geq 5$.

Theorem 1. *Let G be a CHC graph of order $n \geq 5$. If u and v are two adjacent vertices in G, then $d(u) + d(v) \leq n + 2$.*

Proof. Since G is a CHC graph, for any edge uv, there exist two vertices, say $v_1, v_n \in V(G)$ such that edge uv is (v_1, v_n)-critical Hamiltonian edge. Let $P : v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n$ be a Hamiltonian path, say (v_1, v_n)-Hamiltonian path, where $v_i = u$ and $v_{i+1} = v$.

By symmetry, we consider the following three cases.

Case 1. $i = 1$.
Assume that $R = N(v_i) \cap \{v_3, \ldots, v_n\}$, where $|R| = r$ and $S = N(v_{i+1}) \cap \{v_4, \ldots, v_n\}$, where $|S| = s$. It is clear that $d(v_1) = r + 1$ and $d(v_2) = s + 2$. For j, $3 \leq j \leq n - 1$, if $v_j \in R$, then $v_{j+1} \notin S$. Otherwise, the path $v_1, v_j, v_{j-1}, v_{j-2}, \ldots, v_2, v_{j+1}, v_{j+2}, \ldots, v_n$ is a (v_1, v_n)-Hamiltonian path in $G - v_1v_2$; a contradiction. Hence, if $v_n \in R$, then at least $r-1$ elements of $\{v_3, \ldots, v_n\}$ are not in S. If $v_n \notin R$, then at least r elements of $\{v_3, \ldots, v_n\}$ are not in S. Thus,

$$s \leq \max\{(n-3) - r, (n-3) - (r-1)\} = n - r - 2.$$

Finally,

$$d(v_1) + d(v_2) = (r+1) + (s+2) \leq (r+1) + (n-r) = n + 1.$$

Case 2. $i = 2$.
Let $R = N(v_i) \cap \{v_4, \ldots, v_n\}$, where $|R| = r$ and $S = N(v_{i+1}) \cap \{v_5, \ldots, v_n\}$, where $|S| = s$. It is easy to see that $d(v_2) = r + 2$ and $d(v_3) \leq s + 3$. Similar to Case 1, for j, $4 \leq j \leq n-1$, if $v_j \in R$, then $v_{j+1} \notin S$. Now if $v_n \in R$, then at least $r - 1$ elements of $\{v_4, \ldots, v_n\}$ are not in S and if $v_n \notin R$, then at least r elements of $\{v_4, \ldots, v_n\}$ are not in S. Hence,

$$s \leq \max\{(n-4) - r, (n-4) - (r-1)\} = n - r - 3.$$

Thus,

$$d(v_2) + d(v_3) \leq (r+2) + (s+3) \leq (r+2) + (n-r) = n + 2.$$

Case 3. $2 < i \leq n/2$.
Let $R_1 = N(v_i) \cap \{v_1, \ldots, v_{i-2}\}$, where $|R_1| = r_1$, $S_1 = N(v_{i+1}) \cap \{v_1, \ldots, v_{i-1}\}$, where $|S_1| = s_1$, $R_2 = N(v_i) \cap \{v_{i+2}, \ldots, v_n\}$, where $|R_2| = r_2$ and $S_2 = N(v_{i+1}) \cap \{v_{i+3}, \ldots, v_n\}$, where $|S_2| = s_2$. Since, $d(v_i) = r_1 + r_2 + 2$ and $d(v_{i+1}) = s_1 + s_2 + 2$. For k, $1 \leq k \leq i-2$, if $v_k \in R_1$, then $v_{k+1} \notin S_1$. Otherwise, the path $v_1, v_2, \ldots, v_k, v_i, v_{i-1}, v_{i-2}, \ldots, v_{k+1}, v_{i+1}, v_{i+2}, \ldots, v_n$ is a (v_1, v_n)-Hamiltonian path in $G - v_i v_{i+1}$, which is contradiction. Hence, $s_1 \leq i - 1 - r_1$. Similarly, for j, $i+2 \leq j \leq n$, if $v_j \in R_2$, then $v_{j+1} \notin S_2$. Depends on $v_n \in R_2$ or $v_n \notin R_2$, we have

$$s_2 \leq \max\{n - (i+2) - r_2, n - (i+2) - (r_2 - 1)\} = n - r_2 - i - 1.$$

Thus,

$$d(v_i) + d(v_{i+1}) = (r_1 + r_2 + 2) + (s_1 + s_2 + 2) \leq n + 2,$$

as desired. ■

Corollary 1. *In every CHC graph of order $n \geq 4$, $3 \leq \delta(G) \leq \lfloor (n+2)/2 \rfloor$.*

Corollary 2. *Every CHC graph of order $n \geq 5$ has at most one vertex of degree $n - 1$.*

Theorem 2. *A CHC graph G of order $n \geq 5$ has a vertex of degree $n - 1$ if and only if G is a wheel.*

Proof. Let v be the only vertex of degree $n - 1$ in G. Since v is adjacent to all the other vertices, by Theorem 1, the other vertices in G are of degree 3. Hence, every vertex in $G - v$ has degree 2. On the other hand, by Proposition 1, $G - v$ is a 2-connected graph. Therefore, $G - v$ is a cycle of length $n - 1$. Hence, G is a wheel. The converse is trivial. ■

Corollary 3. *For $n = 5$, wheel is the only CHC graph of order n.*

Proof. If G is a CHC graph of order 5, then $3 \leq \delta(G) \leq \Delta(G) \leq 4$. By Theorem 2, if $\Delta(G) = 4$ then G is a wheel. Otherwise G is 3-regular graph of odd order, which is impossible. ∎

As we have shown, there is no CHC graph of order $n \leq 5$ and $\Delta = n - 2$. Thus, it is natural to ask about existence of CHC graphs of order $n > 5$ and $\Delta = n - 2$.

Theorem 3. *There is no CHC graph of order $n \geq 6$ with maximum degree $n-2$.*

Proof. Let G be a Hamiltonian connected graph of order $n \geq 6$ and maximum degree $n - 2$. Also, assume that $V(G) = \{u, v, v_1, \ldots, v_{n-2}\}$, $d(u) = n - 2$ and v be the unique vertex which is not adjacent to u, where $P : u, v_1, \ldots, v_{n-2}, v$ be a Hamiltonian path from u to v. We prove that G contains a spanning Hamiltonian connected subgraph with a non-critical Hamiltonian edge which is incident to u. Hence by Observation 1, G is not a CHC graph.

Since $d(v) \geq 3$, v is adjacent to a vertex v_i, for some i, $2 \leq i \leq n - 3$ (see Figure 4 (a)). Let i is the largest index which vertex v_i is adjacent to v. We consider the following two possibilities.

Case 1. $i = n - 3$, $d(v_{n-2}) \geq 4$, $N(v) = \{v_{i-1}, v_i, v_{i+1}\}$ and $d(v_i) = 4$.

In this case, if $d(v_{n-2}) = 4$ and $v_{n-2} \nleftrightarrow v_{n-4}$, then $\{u, v_{n-4}\}$ is a vertex cut for G, which is a contradiction. Hence, v_{n-2} is adjacent to some v_k, $1 \leq k < n-4$. On the other words, G contains a spanning subgraph, which is shown in Figure 4 (b).

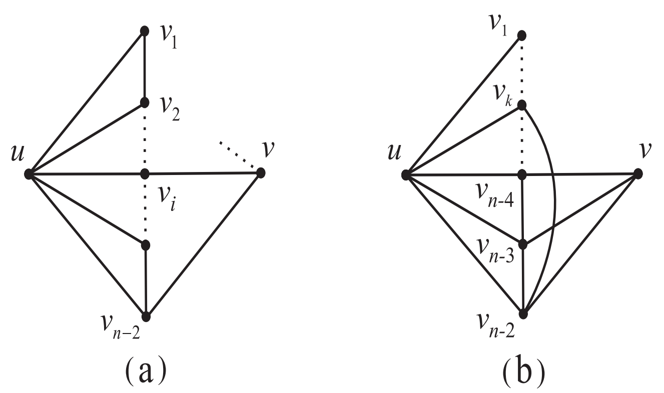

Figure 4: Subgraphs of a Hamiltonian connected graph G with maximum degree $n - 2$.

Now, we prove edge uv_{n-2} is not a critical Hamiltonian edge. For this sake, we assume the path $x_1, x_2, \ldots, (x_{j-1} = u), (x_j = v_{n-2}), x_{j+1}, \ldots, x_n$ is an (x_1, x_n)-Hamiltonian path and show that there exists another Hamiltonian path from x_1 to x_n in $G - uv_{n-2}$.

If $v = x_1$, then there is a vertex $x_l \in N(v_{n-2})$ such that $2 \leq l < j-1$ or $j+1 < l \leq n$. If $l = 2$, then the path $x_1 = v, (x_j = v_{n-2}), x_2, x_3, \ldots, (x_{j-1} = u), x_{j+1}, \ldots, x_n$ is a Hamiltonian path in $G - uv_{n-2}$. If $2 < l < j-1$, then the path $x_1, \ldots, x_{l-1}, (x_{j-1} = u), x_{j-2}, \ldots, x_l, (x_j = v_{n-2}), x_{j+1}, \ldots, x_n$ is a desired Hamiltonian path. If $j+1 < l \leq n$, then the path $x_1, x_2, \ldots, (x_{j-1} = u), x_{l-1}, x_{l-2}, \ldots, (x_j = v_{n-2}), x_l, \ldots, x_n$ is a desired Hamiltonian path.

Now, let $v = x_m$ where, $2 \leq m < j-1$ or $j+1 \leq m \leq n$. If $2 \leq m < j-1$, then the path $x_1, x_2, \ldots, x_{m-1}, (x_{j-1} = u), x_{j-2}, \ldots, (x_m = v), (x_j = v_{n-2}), x_{j+1}, \ldots, x_n$ is a Hamiltonian path which does not contain edge uv_{n-2}. If $m = j+1$, then there is another vertex in $N(v_{n-2})$, say x_l, such that $2 \leq l < j-1$ or $j+1 < l \leq n$. In this case, similar to above we can find a Hamiltonian path in $G - uv_{n-2}$. If $j+1 < m \leq n$, then the path $x_1, x_2, \ldots, (x_{j-1} = u), x_{m-1}, x_{m-2}, \ldots, (x_j = v_{n-2}), (x_m = v), x_{m+1}, \ldots, x_n$ is a desired Hamiltonian path.

Case 2. $i \neq n-3$ or $d(v_{n-2}) = 3$ or $N(v) \neq \{v_{i-1}, v_i, v_{i+1}\}$ or $d(v_i) \neq 4$.

In this case, we claim that edge uv_i is not a critical Hamiltonian edge.

Assume that $P : x_1, x_2, \ldots, x_{j-1}, x_j, \ldots, x_n$ is an arbitrary Hamiltonian path containing uv_i. We provide another Hamiltonian path from x_1 to x_n in $G - uv_i$.

If $v = x_1$, then it is easy to find a Hamiltonian path from v to every other vertices in $G - uv_i$.

If $v = x_m$, where $1 < m < j-1$, then an (x_1, x_n)-Hamiltonian path in $G - uv_i$ is $x_1, x_2, \ldots, x_{m-1}, (x_{j-1} = u), x_{j-2}, \ldots, (x_m = v), (x_j = v_i), x_{j+1}, \ldots, x_n$.
If there is no $1 \leq m < j-1$, such that $v = x_m$, then, if $j+1 < m \leq n$, then the path

$$x_1, x_2, \ldots, (x_{j-1} = u), x_{m-1}, x_{m-2}, \ldots, (x_j = v_i), (x_m = v), x_{m+1}, \ldots, x_n$$

is the desired Hamiltonian path. Otherwise, $m = j+1$. Note that $d(v_i) \geq 4$.

If v_i is adjacent to a vertex x_l, such that $1 < l < j-1$, then the path

$$x_1, x_2, \ldots, x_{l-1}, (x_{j-1} = u), x_{j-2}, \ldots, x_l, (x_j = v_i), (x_{j+1} = v), \ldots, x_n$$

is a Hamiltonian path in $G - uv_i$. If v_i is adjacent to a vertex x_l, such that $j+2 < l \leq n$, then the path

$$x_1, x_2, \ldots, (x_{j-1} = u), x_{l-1}, x_{l-2}, \ldots, (x_{j+1} = v), (x_j = v_i), x_l, x_{l+1}, \ldots, x_n$$

is a Hamiltonian path in $G - uv_i$. If there is no vertex x_l adjacent to v_i, such that $1 < l < j-1$ or $j+2 < l \leq n$, then $N(v_i) = \{x_1, u, v, x_{j+2}\}$. If $j = 3$

and $x_1 \in N(v)$, then the path $x_1, (x_4 = v), (x_3 = v_i), x_5, (x_2 = u), x_6, \dots, x_n$ is a desired Hamiltonian path. If $j \neq 3$, then there is a vertex $x_t \in N(v)$, such that $1 < t < j - 1$ or $j + 2 < t \leq n$. Otherwise, $v_i = v_{n-3}$, $d(v_{n-2}) \geq 4$, $N(v) = \{v_{n-4}, v_{n-3}, v_{n-2}\}$ and $N(v_{n-3}) = \{u, v, v_{n-4}, v_{n-2}\}$, which is a contradiction.

If $1 < t < j - 1$, then the Hamiltonian path

$$x_1, x_2, \dots, x_{t-1}, (x_{j-1} = u), x_{j-2}, \dots, x_t, (x_{j+1} = v), (x_j = v_i), x_{j+2}, x_{j+3}, \dots, x_n$$

does not contain edge uv_i. If $j + 2 < t \leq n$, then the Hamiltonian path

$$x_1, x_2, \dots, (x_{j-1} = u), x_{t-1}, x_{t-2}, \dots, x_{j+2}, (x_j = v_i), (x_{j+1} = v), x_t, x_{t+1}, \dots, x_n$$

does not contain edge uv_i. Clearly, if $x_1 \notin N(v)$, then there is a vertex $x_t \in N(v)$, such that $1 < t < j - 1$ or $j + 2 < t \leq n$, which has been considered above. ■

In the following theorem, we construct CHC graph for some given maximum degree.

Theorem 4. *There exists a CHC graph G of order $n \geq 6$ with $\lceil \frac{n}{2} \rceil \leq \Delta(G) \leq n - 3$.*

Proof. For a given positive integer n, $n \geq 6$, we construct a CHC graph G of order n and maximum degree Δ, where $\lceil \frac{n}{2} \rceil \leq \Delta \leq n - 3$, as follows (see Figure 5).

Let $V(G) = \{u, v, v_1, v_2, \dots, v_{n-2}\}$ and

$$\begin{aligned} E(G) = \quad & \{v_i v_{i+1} : 1 \leq i \leq n-3\} \cup \{uv_1, uv_{n-2}, vv_1, vv_{n-2}\} \\ & \cup \{uv_{2i} : 1 \leq i \leq n-2-\Delta\} \cup \{vv_{2i+1} : 1 \leq i \leq n-2-\Delta\} \\ & \cup \{uv_i : 2n - 2\Delta - 2 \leq i \leq n-3\}. \end{aligned}$$

It can be seen that, there is a Hamiltonian path between every two vertices of G. In Figure 5 as an example, a (v_2, v_4)-Hamiltonian path is shown by bold line. For i, $1 \leq i \leq n-2$, $d(v_i) = 3$. Thus, every edge in G is incident to a vertex of degree 3; hence is a critical Hamiltonian edge. Moreover, $d(u) = \Delta$ and $d(v) = n - \Delta$, where $\Delta \geq \lceil \frac{n-4}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil$. Therefore, G is a desired graph. ■

Theorems 3 and 4 guarantee the existence of CHC graphs of order $n \leq 7$ with a feasible maximum degree. For $n \geq 8$, the existence of a CHC graph with $3 \leq \Delta < \lceil \frac{n}{2} \rceil$ could be a worthwhile question.

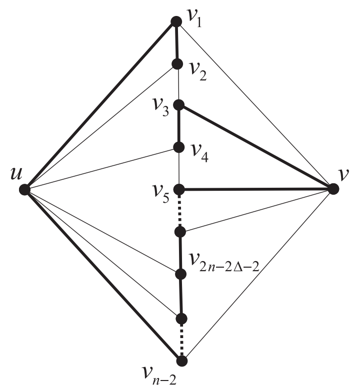

Figure 5: A CHC graph with $\lceil \frac{n}{2} \rceil \leq \Delta \leq n-3$.

3 Hamiltonian connectivity of Harary graphs

In this section, we study Hamiltonian connectivity of Harary graphs. For $k < n$, Harary graphs are denoted by $H_{k,n}$ and obtained by placing n vertices around a circle. If k is even, then each vertex is adjacent to the nearest $k/2$ vertices in both direction clockwise and counterclockwise around the circle. If k is odd and n is even, then each vertex is adjacent to the nearest $(k-1)/2$ vertices in both direction and is adjacent to opposite vertex on the diameter of the circle. If k and n are both odd, then put $V(H_{k,n}) = \{v_0, v_1, \ldots, v_{n-1}\}$ and obtain $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $v_i v_{i+(n-1)/2}$, $0 \leq i \leq (n-1)/2$ [1].

For $n \geq 4$ and $k = 2$, $H_{k,n}$ is a cycle which is not Hamiltonian connected. The next theorems shows Hamiltonian connectivity of $H_{3,n}$ for some integer n.

Theorem 5. *If $n \equiv j \pmod 8$, where $j \in \{0, 4\}$, then $H_{3,n}$ is a CHC graph.*

Proof. For $j = 0$, let $n = 8k$. We find a Hamiltonian path between an arbitrary vertex w_0 and other vertices in $H_{3,n}$. To do this, we divide the outer cycle to four regions such that the vertices w_0, x_0, y_0, z_0 are the boundaries of these regions and w_0 is adjacent to y_0 and x_0 is adjacent to z_0. Let the outer cycle be in the form

$$w_0, w_1, \ldots, w_{2k-1}, x_0, x_1, \ldots, x_{2k-1}, y_0, y_1, \ldots, y_{2k-1}, z_0, z_1, \ldots, z_{2k-1}, w_0$$

(see Figure 6). Hence, for i, $0 \leq i \leq 2k-1$, w_i is adjacent to y_i and x_i is adjacent to z_i. If $i = 2t$, $1 \leq t \leq k-1$, then the path

$$w_{2t}, y_{2t}, y_{2t+1}, w_{2t+1}, w_{2t+2}, y_{2t+2}, y_{2t+3}, \ldots, y_{2k-1}, w_{2k-1}, x_0, z_0, z_1, x_1, x_2, z_2, z_3, \ldots, x_{2k-3}, x_{2k-2}, z_{2k-2}, z_{2k-1}, x_{2k-1}, y_0, y_1, \ldots, y_{2t-2}, y_{2t-1}, w_{2t-1}, \ldots, w_1, w_0$$

and, if $i = 2t$, $0 \leq t \leq k-1$, then the path

$$x_{2t}, x_{2t-1}, \ldots, x_1, x_0, z_0, z_1, \ldots, z_{2t}, z_{2t+1}, x_{2t+1}, x_{2t+2}, z_{2t+2}, z_{2t+3}, \ldots, x_{2k-3}, x_{2k-2}, z_{2k-2}, z_{2k-1}, x_{2k-1}, y_0, y_1, \ldots, y_{2k-1}, w_{2k-1}, w_{2k-2}, \ldots, w_1, w_0$$

are Hamiltonian paths from w_i to w_0 and x_i to w_0, respectively.

If $i = 2t-1$, $1 \le t \le k$, then the paths

$$w_{2t-1}, w_{2t}, \ldots, w_{2k-1}, x_0, x_1, \ldots, x_{2k-1}, z_{2k-1}, \ldots, z_1, z_0, y_{2k-1}, \ldots, y_{2t+1}, \\ y_{2t}, y_{2t-1}, y_{2t-2}, w_{2t-2}, w_{2t-3}, \ldots, y_3, y_2, w_2, w_1, y_1, y_0, w_0$$

and

$$x_{2t-1}, x_{2t}, z_{2t}, z_{2t+1}, x_{2t+1}, x_{2t+2}, z_{2t+2}, z_{2t+3}, \ldots, x_{2k-3}, x_{2k-2}, z_{2k-2}, z_{2k-1}, x_{2k-1}, \\ y_0, y_1, \ldots, y_{2k-1}, z_0, z_1, \ldots, z_{2t-1}, z_{2t-2}, x_{2t-2}, \ldots, x_1, x_0, w_{2k-1}, \ldots, w_1, w_0$$

are Hamiltonian paths from w_i to w_0 and x_i to w_0, respectively.

Finally, the path

$$y_0, y_1, \ldots, y_{2k-1}, z_0, z_1, \ldots, z_{2k-1}, x_{2k-1}, x_{2k-2}, \ldots, x_0, w_{2k-1}, w_{2k-2}, \ldots, w_0$$

is a Hamiltonian path from y_0 to w_0.

Hamiltonian paths from y_i, $1 \le i \le 2k-1$, to w_0 and z_i, $0 \le i \le 2k-1$, to w_0 can be found by symmetry. Recall that, $H_{3,n}$ is 3-regular; hence, is a CHC graph.

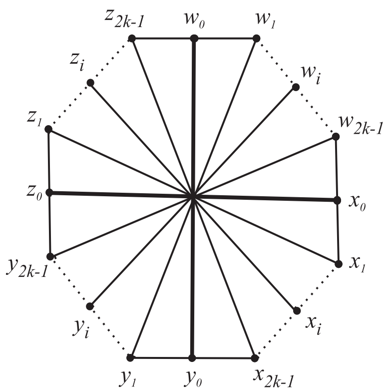

Figure 6: Harary graph $H_{3,n}$, $n \equiv 0 \pmod 8$.

For $j = 4$, let $n = 8k+4$. If $k = 0$, then $H_{3,n} = K_4$ which is a CHC graph. For $k \ge 1$ similar to above, we divide the outer cycle to four regions with boundaries w_0, x_0, y_0, and z_0. Hence, the outer cycle is in the form

$$w_0, w_1, \ldots, w_{2k}, x_0, x_1, \ldots, x_{2k}, y_0, y_1, \ldots, y_{2k}, z_0, z_1, \ldots, z_{2k}, w_0.$$

By symmetry it is enough to find a Hamiltonian path from w_0 to vertices w_i, $1 \le i \le 2k$, x_i, $0 \le i \le 2k$, and y_0.
For $i = 2t$, $1 \le t \le k$, the path

$$w_{2t}, y_{2t}, y_{2t+1}, w_{2t+1}, w_{2t+2}, y_{2t+2}, y_{2t+3}, \ldots, w_{2k}, y_{2k}, z_0, x_0, x_1, z_1, z_2, \\ x_2, x_3, \ldots, z_{2k}, x_{2k}, y_0, y_1, y_2, \ldots, y_{2t-1}, w_{2t-1}, w_{2t-2}, \ldots, w_1, w_0$$

is a Hamiltonian path from w_i to w_0.
For $i = 2t$, $0 \le t \le k$, the path

$$x_{2t}, x_{2t+1}, \ldots, x_{2k}, z_{2k}, z_{2k-1}, \ldots, z_{2t}, z_{2t-1}, x_{2t-1}, x_{2t-2}, \ldots, x_3, x_2, z_2, \\ z_1, x_1, x_0, z_0, y_{2k}, w_{2k}, w_{2k-1}, y_{2k-1}, y_{2k-2}, \ldots, w_2, w_1, y_1, y_0, w_0$$

is a Hamiltonian path from x_i to w_0.
If $i = 2t-1$, $1 \leq t \leq k$, then the paths

$$w_{2t-1}, w_{2t}, \dots, w_{2k}, x_0, x_1, \dots, x_{2k}, z_{2k}, z_{2k-1}, \dots, z_0, y_{2k}, y_{2k-1}, \dots, y_{2t},$$
$$y_{2t-1}, y_{2t-2}, w_{2t-2}, w_{2t-3}, y_{2t-3}, y_{2t-4}, \dots, w_2, w_1, y_1, y_0, w_0$$

and

$$x_{2t-1}, z_{2t-1}, z_{2t}, x_{2t}, x_{2t+1}, z_{2t+1}, z_{2t+2}, \dots, z_{2k-1}, z_{2k}, x_{2k}, y_0, y_1, y_2, \dots, y_{2k},$$
$$z_0, z_1, \dots, z_{2t-2}, x_{2t-2}, x_{2t-3}, \dots, x_0, w_{2k}, w_{2k-1}, \dots, w_1, w_0$$

are Hamiltonian paths from w_i to w_0 and x_i to w_0, respectively.
Finally, a Hamiltonian path from y_0 to w_0 is

$$y_0, y_1, \dots, y_{2k}, z_0, z_1, \dots, z_{2k}, x_{2k}, x_{2k-1}, \dots, x_0, w_{2k}, w_{2k-1}, \dots, w_0,$$

which completes the proof. ■

Theorem 6. *If $n \equiv j \pmod 8$, where $j \in \{1, 5\}$, then $H_{3,n}$ is a CHC graph.*

Proof. Consider the outer cycles which introduced in the proof of Theorem 5. Note that $H_{3,n}$ where $n \equiv j \pmod 8$ and $j = 1$ or $j = 5$ can be obtained from $H_{3,n-1}$ by adding a new vertex z on the outer cycle such that $z \leftrightarrow w_0$, $z \leftrightarrow y_0$ and $z \leftrightarrow w_{2k-1}$ or $z \leftrightarrow w_{2k}$, respectively. Figure 7 shows $H_{3,n}$, where $n \equiv 1 \pmod 8$
By Theorem 5, there is a Hamiltonian path between every two distinct vertices

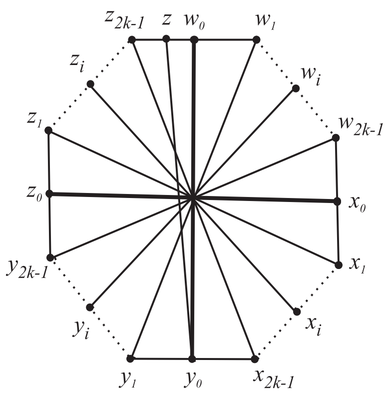

Figure 7: Harary graph $H_{3,n}$, $n \equiv 1 \pmod 8$.

of $H_{3,n-1}$. To complete the proof we need to show that,
1) there is a Hamiltonian path from z to every other vertices of $H_{3,n}$.
2) it is possible to add vertex z to every Hamiltonian path with arbitrary end vertices x_1 and x_{n-1} in $H_{3,n-1}$.

To prove (1), add vertex z to the end of every Hamiltonian path introduced in the proof of Theorem 5. For example, if $n \equiv 1 \pmod 8$ and $1 \leq t \leq k-1$, then a Hamiltonian path between w_{2t} and z is in the form

$w_{2t}, y_{2t}, y_{2t+1}, \ldots, y_{2k-1}, w_{2k-1}, x_0, z_0, z_1, x_1, x_2, z_2, z_3, \ldots, x_{2k-3}, x_{2k-2},$
$z_{2k-2}, z_{2k-1}, x_{2k-1}, y_0, y_1, \ldots, y_{2t-2}, y_{2t-1}, w_{2t-1}, \ldots, w_1, w_0, z.$

We prove (2) for $n \equiv 1 \pmod 8$. The case for $n \equiv 5 \pmod 8$ is similar. Consider an arbitrary Hamiltonian path $x_1, x_2, \ldots, x_{i-1}, x_i = w_0, x_{i+1}, \ldots, x_{n-1}$ in $H_{3,n-1}$. Note that at least one of x_{i-1} or x_{i+1} belongs to $\{y_0, z_{2k-1}\}$.
If $x_{i-1} \in \{y_0, z_{2k-1}\}$, then the path $x_1, x_2, \ldots, x_{i-1}, z, x_i = w_0, x_{i+1}, \ldots, x_{n-1}$ is a Hamiltonian path in $H_{3,n}$. If $x_{i+1} \in \{y_0, z_{2k-1}\}$, then the path $x_1, x_2, \ldots, x_{i-1}$, $x_i = w_0, z, x_{i+1}, \ldots, x_{n-1}$ is a Hamiltonian path in $H_{3,n}$. ■

By Theorems 5 and 6, Harary graph $H_{3,n}$, where $n \equiv j \pmod 8$ and $j \in \{0, 1, 4, 5\}$ are CHC graphs. But for $j \in \{2, 3, 6, 7\}$, $H_{3,n}$ is not necessarily CHC. Figure 8 shows some Harary graphs which don't have any Hamiltonian path from vertex v to w.

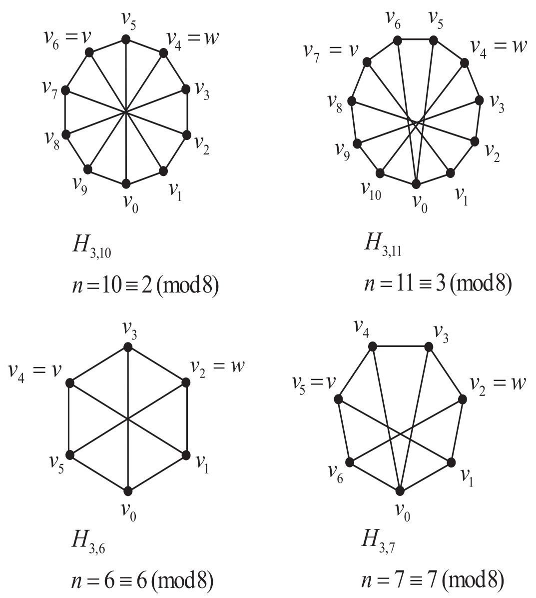

Figure 8: Harary graphs $H_{3,n}$, which are not CHC.

In the following, we prove that $H_{k,n}$ for $k \geq 4$ and $n \geq 5$ is not a critical Hamiltonian connected graph. First we consider $H_{4,n}$.

Theorem 7. *For $n \geq 5$, Harary graph $H_{4,n}$ is Hamiltonian connected.*

Proof. Let $v_1, \ldots, v_n, v_1$ be the outer cycle of $H_{4,n}$. For two distinct vertices v_i and v_j, where $i < j$, consider the following cases.

If $j - i \leq 2$, then $v_i \leftrightarrow v_j$ and $v_{i-1} \leftrightarrow v_{j-1}$. In this case, a Hamiltonian path from v_i to v_j is $v_j, v_{j+1}, \ldots, v_n, v_1, v_2, \ldots, v_{i-1}, v_{j-1}, v_i$.
If $j - i > 2$, then, if for some integer p, $j - i = 2p$, then a Hamiltonian path from v_i to v_j is

$$v_i, v_{i-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{j+1}, v_{j-1}, v_{j-3}, v_{j-5}, \ldots, v_{i+1}, v_{i+2}, v_{i+4}, \ldots, v_{j-2}, v_j.$$

If for some integer p, $j - i = 2p + 1$, then a Hamiltonian path from v_i to v_j is

$$v_i, v_{i-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{j+1}, v_{j-1}, v_{j-3}, v_{j-5}, \ldots, v_{i+2}, v_{i+1}, v_{i+3}, \ldots, v_{j-2}, v_j.$$

■

Note that every edge in Harary graphs is a chord or lies on the outer cycle. In the following theorem, we show that $H_{4,n}$, $n \geq 5$, is not a CHC graph.

Theorem 8. *For $n \geq 5$, every edge on the outer cycle of $H_{4,n}$ is not critical Hamiltonian edge.*

Proof. Let edge xy be on the outer cycle and incident to a Hamiltonian path from vertex v_i to v_j. Without loss of generality, we assume for some l, $i \leq l \leq j$, $x = v_l$ and $y = v_{l+1}$ (otherwise, make an ordering for the vertices on the outer cycle to satisfy this condition). We need to show that, there is a Hamiltonian path from v_i to v_j in $H_{4,n} - xy$. According to this, there are two possibilities.
If $l - i + 1 \leq 2$ and $j - l \leq 2$, then the path

$$v_i, (v_l = x), v_{i-1}, v_{i-2}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{j+1}, (v_{l+1} = y), v_j$$

is a Hamiltonian path from v_i to v_j which does not contain xy.
If $l - i + 1 > 2$, when $n - j + i$ is even, then the path

$$v_j, v_{j-1}, \ldots, v_{l+1}, v_{l-1}, v_l, v_{l-2}, v_{l-3}, \ldots, v_{i+1}, v_{i-1}, v_{i-3}, \ldots, v_{j+3},$$
$$v_{j+1}, v_{j+2}, v_{j+4}, \ldots, v_{i-2}, v_i$$

is a Hamiltonian path from v_j to v_i without using xy, and for $n - j + i$ odd, the path

$$v_j, v_{j-1}, \ldots, v_{l+1}, v_{l-1}, v_l, v_{l-2}, v_{l-3}, \ldots, v_{i+1},$$
$$v_{i-1}, v_{i-3}, \ldots, v_{j+2}, v_{j+1}, v_{j+3}, v_{j+5}, \ldots, v_{i-2}, v_i$$

is a Hamiltonian path from v_j to v_i without using xy.
If $j - l > 2$, then the Hamiltonian path for $n - j + i$ even, is

$$v_i, v_{i+1}, \ldots, v_l, v_{l+2}, v_{l+1}, v_{l+3}, v_{l+4}, \ldots, v_{j-1},$$
$$v_{j+1}, v_{j+3}, \ldots, v_{i-1}, v_{i-2}, v_{i-4}, \ldots, v_{j+2}, v_j$$

and for $n - j + i$ odd, is

$$v_i, v_{i+1}, \ldots, v_l, v_{l+2}, v_{l+1}, v_{l+3}, v_{l+4}, \ldots, v_{j-1},$$
$$v_{j+1}, v_{j+3}, \ldots, v_{i-2}, v_{i-1}, v_{i-3}, v_{i-5}, \ldots, v_{j+2}, v_j.$$

Hence, xy is not a critical Hamiltonian edge. ■

Let $Q_1 : v_i, v_{i+1}, \ldots, v_k$ and $Q_2 : v_{k+1}, v_{k+2}, \ldots, v_j$ be two vertex disjoint paths in connected graph G, where $v_k \leftrightarrow v_{k+1}$. In the following, we mean by $P = Q_1 + Q_2$ the path $P : v_i, v_{i+1}, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_j$

Theorem 9. *For $n \geq 5$, every chord of $H_{4,n}$ is not critical Hamiltonian edge.*

Proof. Let $v_1, v_2, \ldots, v_n, v_1$ be the outer cycle of $H_{4,n}$. We will find two Hamiltonian paths P_1 and P_2 between every two arbitrary vertices v_i and v_j, $1 \leq i < j \leq n$, such that $E(P_1) \cap E(P_2)$ doesn't contain any chord of $H_{4,n}$.

By means of defined paths in Table 1, two Hamiltonian paths P_1 and P_2 may be found (Table 2).

Table 1: Some paths on the outer cycle of $H_{4,n}$.

for every i and j	$Q_1 : v_i, v_{i-1}, \ldots, v_1, v_n, \ldots, v_{j+1}$
$n - j + i$ is even	$Q_1' : v_{i-1}, v_{i-3}, \ldots, v_{j+1}, v_{j+2}, v_{j+4}, \ldots, v_{i-2}, v_i$
$n - j + i$ is odd	$Q_1'' : v_{i-1}, v_{i-3}, \ldots, v_{j+2}, v_{j+1}, v_{j+3}, \ldots, v_{i-2}, v_i$
for every i and j	$Q_2 : v_j, v_{j-1}, v_{j-2}, \ldots, v_{i+1}$
$j - i$ is even	$Q_2' : v_{j-1}, v_{j-3}, \ldots, v_{i+1}, v_{i+2}, v_{i+4}, \ldots, v_{j-2}, v_j$
$j - i$ is odd	$Q_2'' : v_{j-1}, v_{j-3}, \ldots, v_{i+2}, v_{i+1}, v_{i+3}, \ldots, v_{j-2}, v_j$

Table 2: Hamiltonian paths P_1 and P_2 with no common chord.

Both of n and $j - i$ are even	$P_1 = Q_1 + Q_2'$	$P_2 = Q_2 + Q_1'$
Both of n and $j - i$ are odd	$P_1 = Q_1 + Q_2''$	$P_2 = Q_2 + Q_1'$
n is even and $j - i$ is odd	$P_1 = Q_1 + Q_2''$	$P_2 = Q_2 + Q_1''$
n is odd and $j - i$ is even	$P_1 = Q_1 + Q_2'$	$P_2 = Q_2 + Q_1''$

Hence, for every arbitrary chord e in $H_{4,n}$, there exists at least one Hamiltonian path between v_i and v_j, excluding e. ■

Corollary 4. *For every $k \geq 4$ and $n \geq 5$, $H_{k,n}$ is a Hamiltonian connected graph, but it is not a CHC graph. Moreover, every edge of $H_{k,n}$ is not a critical Hamiltonian edge.*

Proof. For $k = 4$, the statement is obtained from Theorems 7, 8 and 9. For $k > 4$, $H_{k,n}$ contains a spanning subgraph $H_{4,n}$. Thus by Observation 1, $H_{k,n}$ is a Hamiltonian connected graph which is not CHC and every edge of $H_{k,n}$ is not critical Hamiltonian edge. ■

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