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Constructing regular graphs with smallest defining number

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Abstract

In a given graph G , a set S of vertices with an assignment of colors is a defining set of the vertex coloring of G , if there exists a unique extension of the colors of S to a $\chi(G)$ -coloring of the vertices of G . A defining set with minimum cardinality is called a smallest defining set (of vertex coloring) and its cardinality, the defining number, is denoted by $d(G, \chi)$. Let $d(n, r, \chi = k)$ be the smallest defining number of all r -regular k -chromatic graphs with n vertices. Mahmoodian et. al [7] proved that, for a given k and for all $n \geq 3k$, if $r \geq 2(k-1)$ then $d(n, r, \chi = k) = k-1$. In this paper we show that for a given k and for all $n < 3k$ and $r \geq 2(k-1)$, $d(n, r, \chi = k) = k-1$.

Keywords: regular graphs, colorings, defining sets, uniquely extendible colorings.

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1 Introduction

A k -coloring of a graph G is an assignment of k different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number, $\chi(G)$, of a graph G is the minimum number k for which there exists a k -coloring for G . A graph G with $\chi(G) = k$ is called a k -chromatic graph. In a given graph G , a set of vertices S with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of S to a $\chi(G)$ -coloring of the vertices of G . A defining set with minimum cardinality is called a smallest defining set (of a vertex coloring) and its cardinality is the defining number, denoted by $d(G, \chi)$.

There are some results on defining numbers in [6] (see also [3], and [4]). Here we study the following concept. Let $d(n, r, \chi = k)$ be the smallest value of $d(G, \chi)$ for all r -regular k -chromatic graphs with n vertices. Note that for any graph G , we have $d(G, \chi) \geq \chi(G) - 1$, therefore $d(n, r, \chi = k) \geq k - 1$. By Brooks' Theorem [2], if G is a connected r -regular k -chromatic graph which is not a complete graph or an odd cycle, then $k \leq r$. For the case of $r = k$, Mahmoodian and Mendelsohn [5] determined the value of $d(n, k, \chi = k)$ for all $k \leq 5$. Mahmoodian and Soltankhah [8] determined this value for $k = 6$ and $k = 7$. Also in [8], for each k , the value of $d(n, k, \chi = k)$ is determined for some congruence classes of n . For the case of $k < r$, it is proved in [5] that, for each n and each $r \geq 4$, we have $d(n, r, \chi = 3) = 2$. The following question is raised in [5]:

Question. *Is it true that for every k , there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geq n_0(k)$ and $r \geq r_0(k)$ we have $d(n, r, \chi = k) = k - 1$?*

Mahmoodian et. al. [7] proved that the answer to this question is positive and that, for a given k and all $n \geq 3k$, if $r \geq 2(k - 1)$ then $d(n, r, \chi = k) = k - 1$.

We show the above statement for $n < 3k$. In fact we prove that:

Theorem. *Let k be a positive integer. For each $n < 3k$, if $r \geq 2(k - 1)$ then $d(n, r, \chi = k) = k - 1$.*

2 Preliminaries

In this section, we state some known results and definitions which will be used in the sequel.

Definition 1 [5]. Let G and H be two graphs, each with a given proper k -coloring say c_G and c_H , (respectively) with k colors. Then the chromatic join of G and H , denoted by $G \overset{\chi}{\vee} H$ is a graph where $V(G \overset{\chi}{\vee} H)$ is $V(G) \cup V(H)$, and $E(G \overset{\chi}{\vee} H)$ is $E(G) \cup E(H)$, together with the set $\{xy \mid x \in V(G), y \in V(H) \text{ such that } c_G(x) \neq c_H(y)\}$.

Theorem A [5]. Let n be a multiple of k , say $n = kl$ ($l \geq 2$); then $d(kl, 2(k-1), \chi = k) = k-1$.

To prove this theorem Mahmoodian and Mendelsohn constructed a $2(k-1)$ -regular k -chromatic graph with $n = kl$ vertices as follows. Let G_1, G_2, \dots, G_l be vertex disjoint graphs such that G_1 and G_l are two copies of K_k and if $l \geq 3$, G_2, \dots, G_{l-1} are copies of \overline{K}_k . Color each G_i with k colors $1, 2, \dots, k$. Then construct a graph G with lk vertices by taking the union of $G_1 \cup G_2 \cup \dots \cup G_l$, and by making a chromatic join between G_i and G_{i+1} ; for $i = 1, 2, \dots, l-1$. This is the desired graph. We denote such a graph by $G_{l(k)}$ and use this construction in Section 3.

Definition 2 [8]. Let G be a k -chromatic graph and let S be a defining set for G . Then a set $F(S)$ of edges is called nonessential edges, if the chromatic number of $G - F(S)$, the graph obtained from G by removing the edges in $F(S)$, is still k , and S is also a defining set for $G - F(S)$.

Remark 1. A necessary condition for the existence of an r -regular k -chromatic graph is $\frac{r}{k-1} \leq \frac{n}{k}$. For, if G is an r -regular k -chromatic graph with n vertices, then each chromatic class in G has at most $n - r$ vertices. Therefore $n \leq k(n - r)$. This implies $\frac{r}{k-1} \leq \frac{n}{k}$. Thus, for $r \geq 2(k-1)$ there are not any graph of order $n < 2k$. Hence when $r \geq 2(k-1)$, it is sufficient to investigate $d(n, r, \chi = k)$ only for $n \geq 2k$. Also it is obvious that n and r cannot be both odd.

For the definitions and notations not defined here we refer the reader to texts, such as [9].

3 Main results

In this section in the following four theorems we prove our main result, which was mentioned at the end of Section 1.

Theorem 1. For each $k \geq 3$ and each $r \geq 2(k-1)$, we have $d(3k-1, r, \chi = k) = k-1$.

Proof. Let $n = 3k - 1$ and $r = 2(k - 1) + t$. By Remark 1 it is obvious that $t \leq k - 2$. First for $t = 0$, we construct a $2(k - 1)$ -regular k -chromatic graph H with n vertices and $d(H, \chi) = k - 1$ as follows. By Theorem A we have $d(3k, 2(k - 1), \chi = k) = k - 1$. In graph $G_{3(l)}$ which was constructed to prove Theorem A, let $V(G_1) = \{u_1, u_2, \dots, u_k\}$, $V(G_2) = \{v_1, v_2, \dots, v_k\}$, and $V(G_3) = \{w_1, w_2, \dots, w_k\}$. Also assume that $c(u_i) = c(v_i) = c(w_i) = i$, for $i = 1, 2, \dots, k$. Note that the set of vertices adjacent to v_k is $N_{G_{3(l)}}(v_k) = \{u_1, \dots, u_{k-1}\} \cup \{w_1, \dots, w_{k-1}\}$. We delete the vertex v_k and join its neighbors in the following manner: we join u_i to w_{i+1} for $i = 1, 2, \dots, k - 2$ and u_{k-1} to w_1 . It can be easily seen that the new graph, say H , is $2(k - 1)$ -regular k -chromatic with $n = 3k - 1$ vertices with a defining set $S = \{u_1, u_2, \dots, u_{k-1}\}$.

Now for $1 \leq t \leq k - 3$, to construct an r -regular k -chromatic graph, we consider the graph H , and we add the edges $u_i w_{i+j+2 \pmod k}$, for $i = 1, \dots, k$ and $j = 1, \dots, t$, to H . Also, in the case of k odd, we add the edges of t mutually disjoint 1-factors of K_{k-1} , and in the case of k even, the edges of $\frac{t}{2}$ mutually disjoint 2-factors of K_{k-1} , on vertex set $\{v_1, \dots, v_{k-1}\}$.

Note that if $t = k - 2$ then such a graph does not exist. For, if G is a graph satisfying such conditions then we know that each chromatic class in G has at most 3 vertices. Since $n = 3k - 1$, G must have $k - 1$ chromatic classes of size 3 and one chromatic class of size 2. And each vertex in a chromatic class of size 3 must be adjacent to all other vertices. This implies that the degree of each vertex in the chromatic class of size 2 is $3(k - 1) = r + 1$, which contradicts the r -regularity of the graph. ■

Example 1. In Figure 1 we show the graph H when $k = 5$ and $r = 8$. The vertices of the defining set are shown by the filled circles.

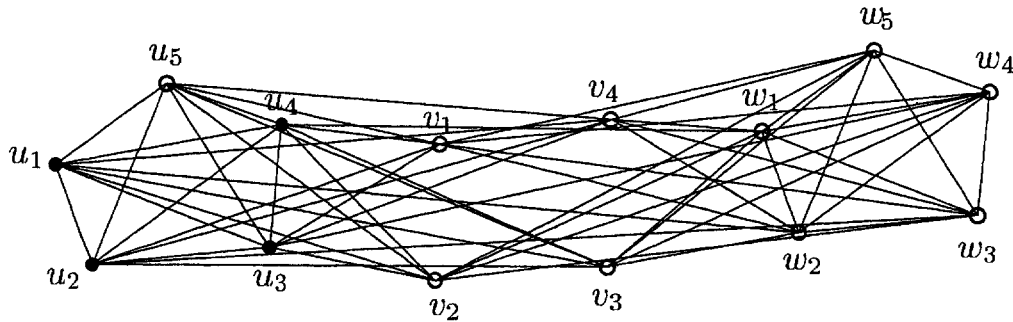


Figure 1: $d(H, \chi = 5) = 4$.

Theorem 2. For each odd number $k \geq 3$, and each $2k \leq n \leq 3k - 2$, we have $d(n, 2(k-1), \chi = k) = k - 1$.

Proof. By Theorem A we have $d(2k, 2(k-1), \chi = k) = k - 1$. Let $n = 2k + s$, $s = 1, 2, \dots, k-2$. We construct a $2(k-1)$ -regular k -chromatic graph H_s with n vertices and defining number equals to $k-1$. For this, we consider graph $G_{2(l)}$ and add s new vertices to it, delete some suitable edges as follows and join the new vertices to the end vertices of deleted edges. In graph $G_{2(l)}$, for convenience let $V(G_1) = \{u_1, \dots, u_i, \dots, u_{\frac{k-1}{2}}, u_{1'}, \dots, u_{i'}, \dots, u_{(\frac{k-1}{2})'}, u_k\}$ and $V(G_2) = \{v_1, \dots, v_i, \dots, v_{\frac{k-1}{2}}, v_{1'}, \dots, v_{i'}, \dots, v_{(\frac{k-1}{2})'}, v_k\}$, where $i' = i + \frac{k-1}{2}$, $i = 1, 2, \dots, \frac{k-1}{2}$; and $c(u_j) = c(v_j) = j$, for $j = 1, 2, \dots, k$.

If $1 \leq s \leq \frac{k-1}{2}$ then denote new vertices by x_1, \dots, x_s . Let $M_1, M_2, \dots, M_{\frac{k-1}{2}}$ be mutually disjoint 1-factors of subgraph $\langle u_1, \dots, u_i, \dots, u_{\frac{k-1}{2}}, u_{1'}, \dots, u_{i'}, \dots, u_{(\frac{k-1}{2})'} \rangle$ in $G_{2(l)}$ such that each edge in M_i has one end in $\{u_1, u_2, \dots, u_{\frac{k-1}{2}}\}$ and the other end in $\{u_{1'}, \dots, u_{(\frac{k-1}{2})'}\}$. For each i ($1 \leq i \leq s$) we join x_i to each of the vertices of M_i , and delete all of the edges of M_i . Also with respect to each $u_a u_b \in M_i$, we delete the edge $u_a u_b$ and join x_i to the vertices u_a and u_b . Now it can be easily seen that $\deg(x_i) = 2(k-1)$. Note that the new graph contains a complete subgraph say, $\langle u_1, u_2, \dots, u_{\frac{k-1}{2}}, v_{1'}, \dots, v_{(\frac{k-1}{2})'}, x_1 \rangle = K_k$ and a defining set $S = \{u_1, \dots, u_{k-1}\}$. Also the colors of vertices of $G_{2(l)}$ force all new vertices to be colored k .

If $\frac{k-1}{2} < s \leq k-2$ then we denote the new vertices by $x_1, x_2, \dots, x_{\frac{k-1}{2}}, y_1, y_2, \dots, y_{s-\frac{k-1}{2}}$. For x_i ($1 \leq i \leq \frac{k-1}{2}$) we proceed as before. For y_t ($1 \leq t \leq s - \frac{k-1}{2}$), first we recognize some nonessential edges in $H_{\frac{k-1}{2}}$. If for each i , we let z_i be either u_i or v_i and, for each j , we let w_j be either u_j or v_j , then the following edges form a nonessential set in $H_{\frac{k-1}{2}}$:

$$F = \{v_i v_j \mid 1 \leq i < j \leq \frac{k-1}{2}\} \cup \{u_{i'} u_{j'} \mid 1' \leq i' < j' \leq (\frac{k-1}{2})'\} \cup \{x_1 u_{i'} \text{ or } x_1 v_i \mid 1 \leq i \leq \frac{k-1}{2}\} \cup \{x_i w_j \mid 2 \leq i \leq \frac{k-1}{2}, 1 \leq j \leq k-1\} \cup \{z_i v_k \mid 1 \leq i \leq k-1\}.$$

There are two cases to be considered.

Case 1. $k = 4l + 1$.

In this case the induced subgraphs $A = \langle u_{1'}, u_{2'}, \dots, u_{(\frac{k-1}{2})'} \rangle$ and $B = \langle v_1, v_2, \dots, v_{\frac{k-1}{2}} \rangle$ are complete graphs $K_{\frac{k-1}{2}}$. So they are 1-factorable. Let $F_1, F_2, \dots, F_{\frac{k-3}{2}}$ and $F'_1, F'_2, \dots, F'_{\frac{k-3}{2}}$ be 1-factorizations of A and B , re-

spectively, such that the edge $u_t' u_{(\frac{k-1}{2})}' \in F_t$ and $v_t v_{\frac{k-1}{2}} \in F_t'$. Now for each t ($1 \leq t \leq s - \frac{k-1}{2} \leq \frac{k-3}{2}$) we delete all of the edges of $F_t \setminus \{u_t' u_{(\frac{k-1}{2})}'\}$ and $F_t' \setminus \{v_t v_{\frac{k-1}{2}}\}$. Also we delete the edges $u_t v_{\frac{k-1}{2}}$ and $u_t' v_k$. Finally we delete all the edges $x_1 v_t, x_2 u_{t+1}, \dots, x_{\frac{k-1}{2}} u_{t+\frac{k-3}{2}} \pmod{\frac{k-1}{2}}$. We join y_t to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $k-1$.

Case 2. $k = 4l + 3$.

In this case the induced subgraphs $A = \langle u_1', u_2', \dots, u_{(\frac{k-1}{2})}', u_k \rangle$ and $B = \langle v_1, v_2, \dots, v_{\frac{k-1}{2}}, v_k \rangle$ are complete graphs $K_{\frac{k+1}{2}}$. Thus they are 1-factorable. Let $F_1, F_2, \dots, F_{\frac{k-1}{2}}$ and $F_1', F_2', \dots, F_{\frac{k-1}{2}}'$ be 1-factorizations of A and B , respectively, such that $u_t' u_k \in F_t$ and $v_t v_k \in F_t'$, for $1 \leq t \leq \frac{k-1}{2}$. Now for each t ($1 \leq t \leq s - \frac{k-1}{2} \leq \frac{k-3}{2}$) we delete all of the edges of $F_t \setminus \{u_t' u_k\}$ and $F_t' \setminus \{v_t v_k\}$. Also we delete the edge $v_k u_t$. Finally we delete the edges $x_1 v_t, x_2 u_{t+1}, \dots, x_{\frac{k-1}{2}} u_{t+\frac{k-3}{2}} \pmod{\frac{k-1}{2}}$. We join y_t to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t + \frac{k-1}{2}$. ■

To illustrate the construction shown in the proof of Theorem 2, we provide the following example.

Example 2. Let $k = 7$. For $15 \leq n \leq 19$, we construct a 12-regular 7-chromatic graph of order n with a defining set of size 6. For $n = 14 + s$, $1 \leq s \leq 5$, we add s new vertices to the 12-regular 7-chromatic graph $G_{2(7)}$ of order 14 and delete some nonessential edges as explained in the proof of Theorem 2.

Table 1: New vertices and deleted edges.

New vertices	x_1	x_2	x_3	y_1	y_2
Deleted edges	$u_1 u_1'$	$u_1 u_2'$	$u_1 u_3'$	$u_2' u_3'$	$u_1' u_3'$
	$u_2 u_2'$	$u_2 u_3'$	$u_2 u_1'$	$v_2 v_3$	$v_1 v_3$
	$u_3 u_3'$	$u_3 u_1'$	$u_3 u_2'$	$x_1 v_1$	$x_1 v_2$
	$v_1 v_1'$	$v_1 v_2'$	$v_1 v_3'$	$x_2 u_2$	$x_2 u_3$
	$v_2 v_2'$	$v_2 v_3'$	$v_2 v_1'$	$x_3 u_3$	$x_3 u_1$
	$v_3 v_3'$	$v_3 v_1'$	$v_3 v_2'$	$v_7 u_1$	$v_7 u_2$

Table 1 gives all the deleted edges of $G_{2(7)}$ with respect to addition of new vertices. In Figure 2, we show the deleted edges and the added edges

to construct a 12-regular 7-chromatic graph H_1 of order 15 ($s = 1$) with a defining set of size 6. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

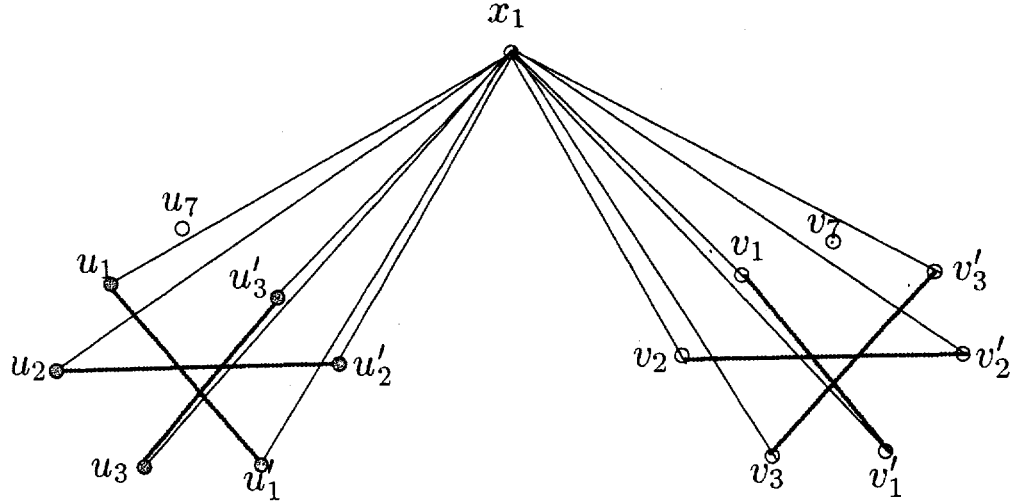


Figure 2: $d(H_1, \chi = 7) = 6$.

Theorem 3. For each even number $k \geq 4$, and each $2k \leq n \leq 3k - 2$, we have $d(n, 2(k - 1), \chi = k) = k - 1$.

Proof. By Theorem A we have $d(2k, 2(k - 1), \chi = k) = k - 1$. For $n = 2k + s$, $s = 1, 2, \dots, k - 2$, we construct a $2(k - 1)$ -regular k -chromatic graph H_s with n vertices and defining number equal to $k - 1$.

To construct H_s , we consider graph $G_{2(k)}$ and add s new vertices to it, delete some suitable edges and join the new vertices to the end vertices of the deleted edges as follows. In graph $G_{2(k)}$ for convenience let $V(G_1) = \{u_1, \dots, u_i, \dots, u_{\frac{k}{2}}, u_{1'}, \dots, u_{i'}, \dots, u_{(\frac{k}{2})'}\}$ and $V(G_2) = \{v_1, \dots, v_i, \dots, v_{\frac{k}{2}}, v_{1'}, \dots, v_{i'}, \dots, v_{(\frac{k}{2})'}\}$, where $i' = i + \frac{k}{2}$, $i = 1, 2, \dots, \frac{k}{2}$; and $c(u_j) = c(v_j) = j$, for $j = 1, 2, \dots, k$.

If $1 \leq s \leq \frac{k}{2} - 1$ then we denote the new vertices by x_1, \dots, x_s . Let $M_1, M_2, \dots, M_{\frac{k}{2}}$ be mutually disjoint 1-factors of the induced subgraph $G_1 = \langle u_1, \dots, u_i, \dots, u_{\frac{k}{2}}, u_{1'}, \dots, u_{i'}, \dots, u_{(\frac{k}{2})'} \rangle$, where, for $i = 1, 2, \dots, \frac{k}{2}$;

$$M_i = \{u_1 u_{i'}, u_2 u_{(i+1)'}, \dots, u_t u_{(i+t-1)'}, \dots, u_{\frac{k}{2}} u_{(i+\frac{k}{2}-1)'}\} \pmod{\frac{k}{2}}.$$

Also let $M'_1, M'_2, \dots, M'_{\frac{k}{2}}$ be mutually disjoint 1-factors of the induced subgraph $G_2 = \langle v_1, \dots, v_i, \dots, v_{\frac{k}{2}}, v'_1, \dots, v'_i, \dots, v_{(\frac{k}{2})}' \rangle$, where, for $i = 1, 2, \dots, \frac{k}{2}$;

$$M'_i = \{v_1 v'_i, v_2 v'_{(i+1)'}, \dots, v_t v'_{(i+t-1)'}, \dots, v_{\frac{k}{2}} v'_{(i+\frac{k}{2}-1)'}\} \pmod{\frac{k}{2}}.$$

Now for each i ($i = 1, 2, \dots, s$) we delete all of the edges of $M_{i+1} \setminus \{u_{\frac{k}{2}-i} u_{(\frac{k}{2})}'\}$, and all of the edges of $M'_i \setminus \{v_{\frac{k}{2}-i+1} v_{(\frac{k}{2})}'\}$. Finally we delete the edge $u_{\frac{k}{2}-i} v_{\frac{k}{2}-i+1}$. We join x_i to the ends of all deleted edges. Now it can be easily seen that $\deg(x_i) = 2(k-1)$. Note that the new graph contains a complete subgraph say $\langle u_1, u_2, \dots, u_{\frac{k}{2}}, u_{(\frac{k}{2})}', v'_1, \dots, v_{(\frac{k}{2}-1)'} \rangle = K_k$ and a defining set $S = \{u_1, \dots, u_{k-1}\}$. Also the colors of vertices of $G_{2(k)}$ force the colors of all new vertices to be k .

If $\frac{k}{2} \leq s \leq k-2$ then we denote the new vertices by $x_1, x_2, \dots, x_{\frac{k}{2}-1}, y_1, y_2, \dots, y_{s-\frac{k}{2}+1}$. For x_i ($1 \leq i \leq \frac{k}{2}-1$) we treat as before. For y_t ($1 \leq t \leq s - \frac{k}{2} + 1$) first we recognize some nonessential edges in $H_{\frac{k}{2}-1}$. If for each j , we let w_j be either u_j or v_j , then the following edges form a nonessential set in $H_{\frac{k}{2}-1}$:

$$F = \{v_i v_j \mid 1 \leq i < j \leq \frac{k}{2}, j \neq i+1\} \cup \{u_{i'} u_{j'} \mid 1' \leq i' < j' \leq (\frac{k}{2})' - 1\} \cup \{x_i w_j \mid 1 \leq i \leq \frac{k}{2}-1, 1 \leq j \leq k-1\} \cup \{v_i v_{(\frac{k}{2})}' \mid 1 \leq i \leq (\frac{k}{2})' - 1\} \cup M_1 \setminus \{u_{\frac{k}{2}} u_{(\frac{k}{2})}'\} \cup M'_{\frac{k}{2}}.$$

There are two cases to be considered.

Case 1. $k = 4l$.

In this case the induced subgraphs $A = \langle u'_1, u'_2, \dots, u_{(\frac{k}{2})}' \rangle$ and $B = \langle v_1, v_2, \dots, v_{\frac{k}{2}} \rangle$ are complete graphs $K_{\frac{k}{2}}$. So they are 1-factorable. Let $F_1, F_2, \dots, F_{\frac{k}{2}-1}$ and $F'_1, F'_2, \dots, F'_{\frac{k}{2}-1}$ be standard 1-factorizations (see [1], page 166) of A and B , respectively, such that the edges $u_t u_{(\frac{k}{2})}' \in F_t$ and $v_t v_{\frac{k}{2}} \in F'_t$. Now for each t ($1 \leq t \leq s - \frac{k}{2} + 1 \leq \frac{k}{2} - 1$) we delete all of the edges of $F_t \setminus \{u_t u_{(\frac{k}{2})}'\}$ and F'_t . Also we delete the edge $v_{(t+1)'} v_{(\frac{k}{2})}' \pmod{(\frac{k}{2}-1)}$. If there exist some edges such as $v_i v_{i+1} \in F'_t$, then instead of these edges we delete the edges $v_i v_{i+1} \in M'_{\frac{k}{2}}$.

Also for an arbitrary index i of such as edges $v_i v_{i+1}$ we delete the edge $v_i v_{(\frac{k}{2})}'$ instead of the edge $v_{(t+1)'} v_{(\frac{k}{2})}'$. Finally we delete the edges $x_1 u_{t+1}, x_2 u_{t+2}, \dots, x_{\frac{k}{2}-1} u_{t+\frac{k}{2}-1} \pmod{\frac{k}{2}}$.

We join y_t to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t + \frac{k}{2}$, for $t \neq \frac{k}{2} - 1$ and the color of $y_{\frac{k}{2}-1}$ to be $\frac{k}{2} - 1$.

Case 2. $k = 4l + 2$.

In this case the induced subgraphs $A = \langle u_1, u_2, \dots, u_{(\frac{k}{2})'}, u_1 \rangle$ and $B = \langle v_1, v_2, \dots, v_{\frac{k}{2}}, v_{(\frac{k}{2})'} \rangle$ are complete graphs $K_{\frac{k}{2}+1}$. So they are 1-factorable. Let $F_1, F_2, \dots, F_{\frac{k}{2}}$ and $F'_1, F'_2, \dots, F'_{\frac{k}{2}}$ be 1-factorizations of A and B , respectively, such that $u_1 u_t \in F_t$ and $v_t v_{(\frac{k}{2})'} \in F'_t$. Now for each t ($1 \leq t \leq s - \frac{k}{2} + 1 \leq \frac{k}{2} - 1$) we delete all of the edges of $F_t \setminus \{u_1 u_t, u_{j'} u_{(\frac{k}{2})'}\}$ and F'_t . Also we delete the edge $u_j u_{j'} \in M_1$. If there exist some edges such as $v_i v_{i+1} \in F'_t$ then instead of the edges $v_i v_{i+1}$ we delete the edges $v_i v_{i+1} \in M'_{\frac{k}{2}}$. Finally we delete the edges $x_1 u_{j+1}, x_2 u_{j+2}, \dots, x_{\frac{k}{2}-1} u_{j+\frac{k}{2}-1} \pmod{\frac{k}{2}}$. We join y_t to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of y_t is forced to be $t + \frac{k}{2}$. ■

To illustrate the construction shown in the proof of Theorem 3, we provide the following example.

Example 3. Let $k = 8$. For $17 \leq n \leq 22$, we construct a 14-regular 8-chromatic graph of order n with a defining set of size 7. For $n = 16 + s$, $1 \leq s \leq 6$, we add s new vertices to the 14-regular 8-chromatic graph $G_{2(8)}$ of order 16 and delete some nonessential edges as explained in the proof of Theorem 3.

Table 2: New vertices and deleted edges.

New vertices	x_1	x_2	x_3	y_1	y_2	y_3
Deleted edges	$u_1 u_2'$	$u_1 u_3'$	$u_2 u_1'$	$u_2' u_3'$	$u_1' u_3'$	$u_1' u_2'$
	$u_2 u_3'$	$u_3 u_1'$	$u_3 u_2'$	$v_1 v_4$	$v_2 v_4$	$v_3' v_4$
	$u_4 u_1'$	$u_4 u_2'$	$u_4 u_3'$	$v_2' v_3$	$v_1 v_3$	$v_1' v_2$
	$v_1 v_1'$	$v_1 v_2'$	$v_1 v_3'$	$v_2 v_4'$	$v_3' v_4'$	$v_1 v_4'$
	$v_2 v_2'$	$v_2 v_3'$	$v_3 v_1'$	$x_1 u_2$	$x_1 u_3$	$x_1 u_4$
	$v_3 v_3'$	$v_4 v_1'$	$v_4 v_2'$	$x_2 u_3$	$x_2 u_4$	$x_2 u_1$
	$u_3 v_4$	$u_2 v_3$	$u_1 v_2$	$x_3 u_4$	$x_3 u_1$	$x_3 u_2$

Table 2 gives all the deleted edges of $G_{2(8)}$ with respect to addition of new vertices. In Figure 3, we show the deleted edges and the added edges

to construct a 14-regular 8-chromatic graph H_1 of order 17 ($s = 1$) with a defining set of size 7. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

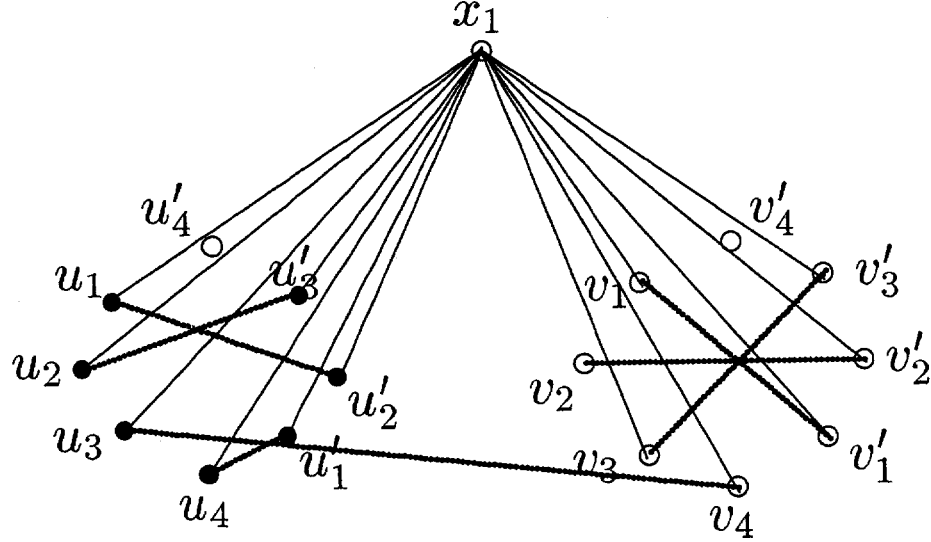


Figure 3: $d(H_1, \chi = 8) = 7$.

Theorem 4. For each $k \geq 4$, $2k \leq n \leq 3k - 2$, and $r > 2(k - 1)$, we have

$$d(n, r, \chi = k) = k - 1.$$

Proof. Let $n = 2k + s$, $0 \leq s \leq k - 2$, and $r = 2(k - 1) + t$. By Remark 1, if there exists an r -regular k -chromatic graph with n vertices then it is obvious that $t < s$. We construct an r -regular k -chromatic graph H with n vertices in the following manner.

Consider graph $G_{2(k)}$, let $V(G_1) = \{u_1, \dots, u_k\}$ and $V(G_2) = \{v_1, \dots, v_k\}$, and $c(u_i) = c(v_i) = i$, for $i = 1, 2, \dots, k$. We add s new vertices say x_1, \dots, x_s to $G_{2(k)}$. For each x_i ($1 \leq i \leq s$) we join x_i to each vertex of $V(G_1) \cup V(G_2) \setminus \{u_i, v_i\}$. Also, in the case of s even, we add the edges of t mutually disjoint 1-factors of K_s , and in the case of s odd, the edges of $\frac{t}{2}$ mutually disjoint 2-factors of K_s , to x_1, \dots, x_s . The graph obtained in this way, say H' , is a k -chromatic graph with n vertices and a defining set $S = \{x_2, \dots, x_s, v_{s+1}, \dots, v_k\}$ such that $\deg(x_i) = 2(k - 1) + t$ ($1 \leq i \leq s$), $\deg(u_i) = \deg(v_i) = 2(k - 1) + s - 1$ ($1 \leq i \leq s$), and $\deg(u_i) = \deg(v_i) = 2(k - 1) + s$ ($s + 1 \leq i \leq k$). Now we show that by deleting some suitable nonessential edges of H' the desired r -regular graph H can be obtained.

In the graph H' , for convenience let $A = \{u_1, \dots, u_{\lfloor \frac{s}{2} \rfloor}\}$, $C = \{u_{\lfloor \frac{s}{2} \rfloor + 1}, \dots, u_s\}$,

1) with
and the

$D = \{u_{s+1}, \dots, u_{s+\lfloor \frac{k-s}{2} \rfloor}\}$, and $B = \{u_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \dots, u_k\}$. Also let $A' = \{v_1, \dots, v_{\lfloor \frac{s}{2} \rfloor}\}$, $C' = \{v_{\lfloor \frac{s}{2} \rfloor+1}, \dots, v_s\}$, $D' = \{v_{s+1}, \dots, v_{s+\lfloor \frac{k-s}{2} \rfloor}\}$, and $B' = \{v_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \dots, v_k\}$. Let $i' = i + \lfloor \frac{k-s}{2} \rfloor$ for $s+1 \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor$.

First we delete a maximal matching of each complete bipartite subgraph with parts B and D of G_1 and parts B' and D' of G_2 . For $k-s$ odd, we assume u_{k-1} and v_k to be vertices unsaturated by the maximal matchings. Then we delete the edge $u_{k-1}v_k$.

Secondly, we delete the edges of $s-t-1$ mutually disjoint maximal matchings of each complete bipartite subgraph with parts $A \cup B$ and $C \cup D$ of G_1 and parts $A' \cup B'$ and $C' \cup D'$ of G_2 . For k odd, we assume that the following vertices are unsaturated by the maximal matchings: $\{u_1, \dots, u_{\lfloor \frac{s}{2} \rfloor}, u_{(s+1)', \dots, u_{(s+1)'+s-t-2-\lfloor \frac{s}{2} \rfloor}}\}$ and $\{v_2, \dots, v_{\lfloor \frac{s}{2} \rfloor}, v_1, v_{(s+1)'+1}, \dots, v_{(s+1)'+s-t-1-\lfloor \frac{s}{2} \rfloor}\}$, in the case of s even, or $\{u_{\lfloor \frac{s}{2} \rfloor+1}, \dots, u_{\lfloor \frac{s}{2} \rfloor+s-t-1}\}$ and $\{v_{\lfloor \frac{s}{2} \rfloor+2}, \dots, v_s, v_{\lfloor \frac{s}{2} \rfloor+1}, v_{s+2}, \dots, v_{2s-t-1-\lfloor \frac{s}{2} \rfloor}\}$, in the case of s odd. Then we delete the edges $u_1v_2, u_2v_3, \dots, u_{\lfloor \frac{s}{2} \rfloor-1}v_{\lfloor \frac{s}{2} \rfloor}, u_{\lfloor \frac{s}{2} \rfloor}v_1, u_{(s+1)'}v_{(s+1)'+1}, \dots, u_{(s+1)'+s-t-2-\lfloor \frac{s}{2} \rfloor}v_{(s+1)'+s-t-1-\lfloor \frac{s}{2} \rfloor}$, or the edges $u_{\lfloor \frac{s}{2} \rfloor+1}v_{\lfloor \frac{s}{2} \rfloor+2}, \dots, u_sv_{\lfloor \frac{s}{2} \rfloor+1}, u_{s+1}v_{s+2}, \dots, u_{\lfloor \frac{s}{2} \rfloor+s-t-1}v_{2s-t-1-\lfloor \frac{s}{2} \rfloor}$, depending on the parity of s , respectively.

If $s-t > \lfloor \frac{k}{2} \rfloor$ then in the second step we delete $\lfloor \frac{k}{2} \rfloor - 1$ maximal matchings. Finally we delete the edges of $s-t-\lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors F_j ($1 \leq j \leq s-t-\lfloor \frac{k}{2} \rfloor$) of bipartite subgraph with parts $C \cup D$ and $C' \cup D'$, where

$$F_j = \{u_i v_{i+j+1} \mid \lfloor \frac{s}{2} \rfloor + 1 \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor - j - 1\} \cup \{u_i v_{i-i_0+\lfloor \frac{s}{2} \rfloor+1} \mid i_0 = s + \lfloor \frac{k-s}{2} \rfloor - j \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor\}.$$

In fact if we consider the order $u_{\lfloor \frac{s}{2} \rfloor+1}, \dots, u_s, u_{s+1}, \dots, u_{s+\lfloor \frac{k-s}{2} \rfloor}$, and $v_{\lfloor \frac{s}{2} \rfloor+1}, \dots, v_s, v_{s+1}, \dots, v_{s+\lfloor \frac{k-s}{2} \rfloor}$, for the vertices in $C \cup D$ and $C' \cup D'$, respectively, then each 1-factor F_j contains the edges in which the i th vertex in $C \cup D$ is matched with $(i+j+1)$ th vertex (mod $|C \cup D|$) in $C' \cup D'$. (See Figure 4.)

Also for decreasing the degree of vertex sets $A \cup B$ and $A' \cup B'$, we delete the edges of $s-t-\lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors F'_j ($1 \leq j \leq s-t-\lfloor \frac{k}{2} \rfloor$) of bipartite subgraph with parts $A \cup B$ and $A' \cup B'$ the same as above. Therefore the graph H obtained in this way contains a complete subgraph say $K_k = \langle A \cup B \cup C' \cup D' \rangle$ and H is an r -regular graph.

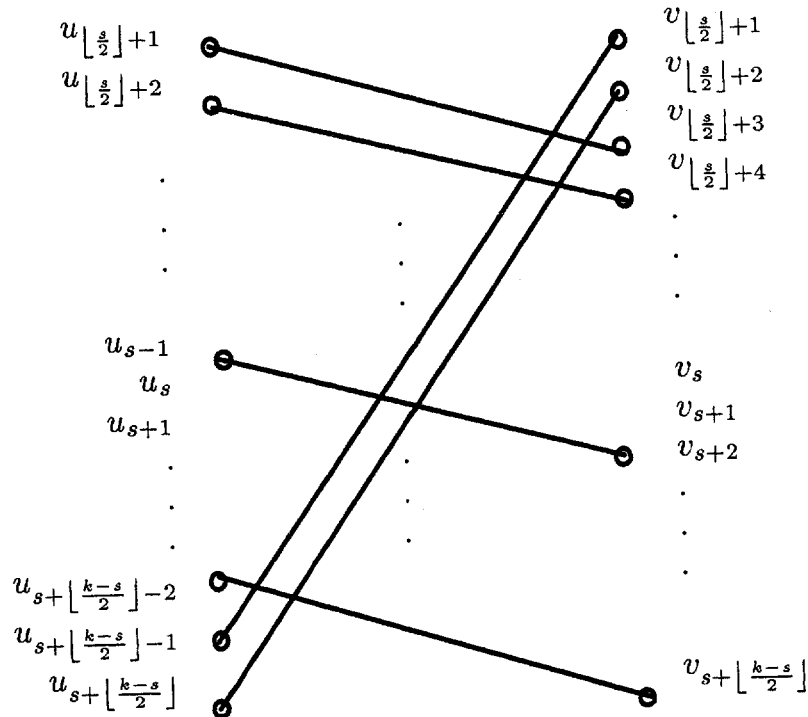


Figure 4: 1-factor F_1 .

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