Characterization of Randomly k-Dimensional Graphs

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Abstract

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. A minimum resolving set for G is a basis of G and its cardinality is the metric dimension of G. The resolving number of a connected graph G is the minimum k, such that every k-set of vertices of G is a resolving set. A connected graph G is called randomly k-dimensional if each k-set of vertices of G is a basis. In this paper, along with some properties of randomly k-dimensional graphs, we prove that a connected graph G with at least two vertices is randomly k-dimensional if and only if G is complete graph K_{k+1} or an odd cycle.

Keywords: Resolving set; Metric dimension; Basis; Resolving number; Basis number; Randomly k-dimensional graph.

1 Preliminaries

In this section, we present some definitions and known results which are necessary to prove our main theorems. Throughout this paper, $G = (V, E)$ is a finite, simple, and connected graph with $e(G)$ edges. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v

in G. The *eccentricity* of a vertex $v \in V(G)$ is $e(v) = \max_{u \in V(G)} d(u, v)$ and the *diameter* of G is $\max_{v \in V(G)} e(v)$. We use $\Gamma_i(v)$ for the set of all vertices $u \in V(G)$ with $d(u, v) = i$. Also, $N_G(v)$ is the set of all neighbors of vertex v in G and $\deg_G(v) = |N_G(v)|$ is the *degree* of vertex v. For a set $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$. If G is clear from the context, it is customary to write $N(v)$ and $\deg(v)$ rather than $N_G(v)$ and $\deg_G(v)$, respectively. The *maximum degree* and *minimum degree* of G, are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a subset S of $V(G)$, $G \setminus S$ is the induced subgraph $\langle V(G) \setminus S \rangle$ of G. A set $S \subseteq V(G)$ is a *separating set* in G if $G \setminus S$ has at least two components. Also, a set $T \subseteq E(G)$ is an *edge cut* in G if $G \setminus T$ has at least two components. A graph G is k-(edge-)connected if the minimum size of a separating set (edge cut) in G is at least k. We mean by $\omega(G)$, the number of vertices in a maximum clique in G. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relations between u and v, respectively. The symbols $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$ represent a path of order n, P_n, and a cycle of order n, C_n, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the (*metric*) *representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. In this case, we say set W resolves G. To see whether a given set W is a resolving set for G, it is sufficient to look at the representations of vertices in $V(G) \backslash W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$. The concepts of resolving sets and metric dimension of a graph are introduced independently by Slater [15] and Harary and Melter [10]. For more results related to these concepts see [1, 2, 3, 5, 9, 13, 14].

We say an ordered set W *resolves* a set T of vertices in G, if the representations of vertices in T are distinct with respect to W. When $W = \{x\}$, we say that vertex x resolves T. The following simple result is very useful.

Observation 1. [11] *Suppose that u, v are vertices in G such that $N(v) \backslash \{u\} =$*

$N(u)\backslash\{v\}$ and W resolves G. Then u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves G.

Let G be a graph of order n. It is obvious that $1 \le \beta(G) \le n-1$. The following theorem characterize all graphs G with $\beta(G) = 1$ and $\beta(G) = n-1$.

Theorem A. [4] Let G be a graph of order n. Then,

(i) $\beta(G) = 1$ if and only if $G = P_n$,

(ii) $\beta(G) = n-1$ if and only if $G = K_n$.

The *basis number* of G, $bas(G)$, is the largest integer r such that every r-set of vertices of G is a subset of some basis of G. Also, the *resolving number* of G, $res(G)$, is the minimum k such that every k-set of vertices of G is a resolving set for G. These parameters are introduced in [6] and [7], respectively. Clearly, if G is a graph of order n, then $0 \le bas(G) \le \beta(G)$ and $\beta(G) \le res(G) \le n-1$. Chartrand et al. [6] considered graphs G with $bas(G) = \beta(G)$. They called these graphs *randomly k-dimensional*, where $k = \beta(G)$. Obviously, $bas(G) = \beta(G)$ if and only if $res(G) = \beta(G)$. In other words, a graph G is randomly k-dimensional if each k-set of vertices of G is a basis of G.

The following properties of randomly k-dimensional graphs are proved in [12].

Proposition A. [12] If $G \neq K_n$ is a randomly k-dimensional graph, then for each pair of vertices $u, v \in V(G)$, $N(v)\backslash\{u\} \neq N(u)\backslash\{v\}$.

Theorem B. [12] If $k \ge 2$, then every randomly k-dimensional graph is 2-connected.

Theorem C. [12] If G is a randomly k-dimensional graph and T is a separating set of G with $|T| = k-1$, then $G \setminus T$ has exactly two components. Moreover, for each pair of vertices $u, v \in V(G) \setminus T$ with $r(u|T) = r(v|T)$, u and v belong to different components.

Theorem D. [12] If $res(G) = k$, then each two vertices of G have at most $k-1$ common neighbors.

Chartrand et al. in [6] characterized the randomly 2-dimensional graphs and proved that a graph G is randomly 2-dimensional if and only if G is an odd cycle. Furthermore, they provided the following question.

Question A. [6] Are there randomly k-dimensional graphs other than complete graph and odd cycles?

In this paper we answer Question A in the negative and prove that G is randomly k-dimensional, $k \geq 3$ if and only if $G = K_{k+1}$.

2 Some Properties of Randomly k-Dimensional Graphs

Let V_p denote the collection of all $\binom{n}{2}$ pairs of vertices of G. Currie and Oellermann [8] defined the *resolving graph* $R(G)$ of G as a bipartite graph with bipartition $(V(G), V_p)$, where a vertex $v \in V(G)$ is adjacent to a pair $\{x, y\} \in V_p$ if and only if v resolves $\{x, y\}$ in G. Thus, the minimum cardinality of a subset S of $V(G)$, where $N_{R(G)}(S) = V_p$ is the metric dimension of G.

In the following through some propositions and lemmas, we prove that if G is a randomly k-dimensional graph of order n and diameter d, then $k \geq \frac{n-1}{d}$.

Proposition 1. *If G is a randomly k-dimensional graph of order n, then*

$$\binom{n}{2}(n-k+1) \leq e(R(G)) \leq n(\binom{n}{2} - k + 1).$$

Proof. Let $z \in V_p$ and $S = \{v \in V(G) \,|\, v \nsim z\}$. Thus, $N_{R(G)}(S) \neq V_p$ and hence, S is not a resolving set for G. If $\deg_{R(G)}(z) \leq n-k$, then $|S| \geq k$, which contradicts $res(G) = k$. Therefore, $\deg_{R(G)}(z) \geq n-k+1$ and consequently, $e(R(G)) \geq \binom{n}{2}(n-k+1)$.

Now, let $v \in V(G)$. If $\deg_{R(G)}(v) \geq \binom{n}{2} - k + 2$, then there are at most $k-2$ vertices in V_p which are not adjacent to v. Let $V_p \backslash N_{R(G)}(v) = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_t, v_t\}\}$, where $t \leq k-2$. Note that, $u_i \sim \{u_i, v_i\}$ in $R(G)$ for each i, $1 \leq i \leq t$. Therefore, $N_{R(G)}(\{v, u_1, u_2, \ldots, u_t\}) = V_p$. Hence, $\beta(G) \leq t+1 \leq k-1$, which is a contradiction. Thus, $\deg_{R(G)}(v) \leq \binom{n}{2} - k + 1$ and consequently, $e(R(G)) \leq n(\binom{n}{2} - k + 1)$. ■

Proposition 2. *If G is a randomly k-dimensional graph of order n, then for each $v \in V(G)$,*

$$\deg_{R(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}.$$

Proof. Note that, a vertex $v \in V(G)$ resolves a pair $\{x, y\}$ if and only if there exist $0 \leq i \neq j \leq e(v)$ such that $x \in \Gamma_i(v)$ and $y \in \Gamma_j(v)$. Therefore, a vertex $\{u, w\} \in V_p$ is not adjacent to v in $R(G)$ if and only if there exists an i, $1 \leq i \leq e(v)$, such that $u, w \in \Gamma_i(v)$. The number of such vertices in V_p is $\sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}$. Therefore, $\deg_{R(G)}(v) = \binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}$. ■

Since $R(G)$ is bipartite, by Proposition 2,

$$e(R(G)) = \sum_{v \in V(G)} [\binom{n}{2} - \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}] = n\binom{n}{2} - \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}.$$

Thus, by Proposition 1,

$$n(k-1) \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2} \leq \binom{n}{2}(k-1). \quad (1)$$

Observation 2. *Let $n_1, ..., n_r$ and n be positive integers, with $\sum_{i=1}^{r} n_i = n$. Then, $\sum_{i=1}^{r} \binom{n_i}{2}$ is minimum if and only if $|n_i - n_j| \leq 1$, for each $1 \leq i, j \leq r$.*

Lemma 1. *Let $n, p_1, p_2, q_1, q_2, r_1$ and r_2 be positive integers, such that $n = p_i q_i + r_i$ and $r_i < p_i$, for $1 \leq i \leq 2$. If $p_1 < p_2$, then*

$$(p_1 - r_1)\binom{q_1}{2} + r_1\binom{q_1+1}{2} \geq (p_2 - r_2)\binom{q_2}{2} + r_2\binom{q_2+1}{2}.$$

Proof. Let $f(p_i) = (p_i - r_i)\binom{q_i}{2} + r_i\binom{q_i+1}{2}$, $1 \leq i \leq 2$. We just need to prove that $f(p_1) \geq f(p_2)$.

$$f(p_1) - f(p_2) = \frac{1}{2}[(p_1 - r_1)q_1(q_1 - 1) + r_1 q_1(q_1 + 1) -$$

$$
\begin{aligned}
& (p_2 - r_2)q_2(q_2-1) - r_2q_2(q_2+1)] \\
= \; & \frac{1}{2}q_1[p_1q_1 - p_1 + 2r_1] - \frac{1}{2}q_2[p_2q_2 - p_2 + 2r_2] \\
= \; & \frac{1}{2}q_1[n - p_1 + r_1] - \frac{1}{2}q_2[n - p_2 + r_2] \\
= \; & \frac{1}{2}[n(q_1 - q_2) - p_1q_1 + r_1q_1 + p_2q_2 - r_2q_2].
\end{aligned}
$$

Since $p_1 < p_2$, we have $q_2 \leq q_1$. If $q_1 = q_2$, then $r_2 < r_1$. Therefore,

$$f(p_1) - f(p_2) = \frac{1}{2}q_1[(p_2 - p_1) + (r_1 - r_2)] \geq 0.$$

If $q_2 < q_1$, then $q_1 - q_2 \geq 1$. Thus,

$$f(p_1) - f(p_2) \geq \frac{1}{2}[n - p_1q_1 + r_1q_1 + q_2(p_2 - r_2)] = \frac{1}{2}[r_1 + r_1q_1 + q_2(p_2 - r_2)] \geq 0.$$

■

Theorem 1. *If G is a randomly k-dimensional graph of order n and diameter d, then $k \geq \dfrac{n-1}{d}$.*

Proof. Note that, for each $v \in V(G)$, $|\bigcup_{i=1}^{e(v)} \Gamma_i(v)| = n - 1$. For $v \in V(G)$, let $n - 1 = q(v)e(v) + r(v)$, where $0 \leq r(v) < e(v)$. Then, by Observation 2,

$$(e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2} \leq \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2}. \tag{2}$$

Let $w \in V(G)$ with $e(w) = d$, $r(w) = r$, and $q(w) = q$, then $n - 1 = qd + r$. Since for each $v \in V(G)$, $e(v) \leq e(w)$, by Lemma 1,

$$(d - r)\binom{q}{2} + r\binom{q+1}{2} \leq (e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}.$$

Therefore,

$$n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} [(e(v) - r(v))\binom{q(v)}{2} + r(v)\binom{q(v)+1}{2}].$$

Thus, by Relations (2) and (1),

$$n[(d-r)\binom{q}{2} + r\binom{q+1}{2}] \leq \sum_{v \in V(G)} \sum_{i=1}^{e(v)} \binom{|\Gamma_i(v)|}{2} \leq \binom{n}{2}(k-1).$$

Hence, $q[(d-r)(q-1)+r(q+1)] \leq (n-1)(k-1)$, which implies, $q[(r-d)+(d-r)q+r(q+1)] \leq (n-1)(k-1)$. Therefore, $q(r-d)+q(n-1) \leq (n-1)(k-1)$. Since $q = \lfloor \frac{n-1}{d} \rfloor$, we have

$$\begin{aligned} k-1 \geq q + q\frac{r-d}{n-1} &= q + \frac{qr}{n-1} - \frac{qd}{n-1} \\ &= q + \frac{qr}{n-1} - \frac{\lfloor \frac{n-1}{d} \rfloor d}{n-1} \\ &\geq q + \frac{qr}{n-1} - 1. \end{aligned}$$

Thus, $k \geq \lfloor \frac{n-1}{d} \rfloor + \frac{qr}{n-1}$. Note that, $\frac{qr}{n-1} \geq 0$. If $\frac{qr}{n-1} > 0$, then $k \geq \lceil \frac{n-1}{d} \rceil$, since k is an integer. If $\frac{qr}{n-1} = 0$, then $r = 0$ and consequently, d divides $n-1$. Thus, $\lfloor \frac{n-1}{d} \rfloor = \lceil \frac{n-1}{d} \rceil$. Therefore, $k \geq \lceil \frac{n-1}{d} \rceil \geq \frac{n-1}{d}$. ■

The following theorem shows that there is no randomly k-dimensional graph of order n, where $4 \leq k \leq n-2$.

Theorem 2. *If G is a randomly k-dimensional graph of order n, then $k \leq 3$ or $k \geq n-1$.*

Proof. For each $W \subseteq V(G)$, let $\overline{N}(W) = V_p \setminus N(W)$ in $R(G)$. We claim that, if $S, T \subseteq V(G)$ with $|S| = |T| = k-1$ and $T \neq S$, then $\overline{N}(S) \cap \overline{N}(T) = \emptyset$. Otherwise, there exists a pair $\{x, y\} \in \overline{N}(S) \cap \overline{N}(T)$. Therefore, $\{x, y\} \notin N(S \cup T)$ and hence, $S \cup T$ is not a resolving set for G. Since $S \neq T$, $|S \cup T| > |S| = k-1$, which contradicts $res(G) = k$. Thus, $\overline{N}(S) \cap \overline{N}(T) = \emptyset$.

Since $\beta(G) = k$, for each $S \subseteq V(G)$ with $|S| = k-1$, $\overline{N}(S) \neq \emptyset$. Now, let $\Omega = \{S \subseteq V(G) \,|\, |S| = k-1\}$. Therefore,

$$|\bigcup_{S \in \Omega} \overline{N}(S)| = \sum_{S \in \Omega} |\overline{N}(S)| \geq \sum_{S \in \Omega} 1 = \binom{n}{k-1}.$$

On the other hand, $\bigcup_{S \in \Omega} \overline{N}(S) \subseteq V_p$. Hence, $|\bigcup_{S \in \Omega} \overline{N}(S)| \leq \binom{n}{2}$. Consequently, $\binom{n}{k-1} \leq \binom{n}{2}$. If $n \leq 4$, then $k \leq 3$. Now, let $n \geq 5$. Thus, $2 \leq \frac{n+1}{2}$. We know that for each $a, b \leq \frac{n+1}{2}$, $\binom{n}{a} \leq \binom{n}{b}$ if and only if $a \leq b$. Therefore, if $k-1 \leq \frac{n+1}{2}$, then $k-1 \leq 2$, which implies $k \leq 3$. If $k-1 \geq \frac{n+1}{2}$, then $n-k+1 \leq \frac{n+1}{2}$. Since $\binom{n}{n-k+1} = \binom{n}{k-1}$, we have $\binom{n}{n-k+1} \leq \binom{n}{2}$ and consequently, $n-k+1 \leq 2$, which yields $k \geq n-1$. ■

By Theorem 2, to characterize all randomly k-dimensional graphs, we only need to consider graphs of order $k+1$ and graphs with metric dimension less than 4. By Theorem A, if G has $k+1$ vertices and $\beta(G) = k$, then $G = K_{k+1}$. Also, if $k = 1$, then $G = P_n$. Clearly, the only paths with resolving number 1 are $P_1 = K_1$ and $P_2 = K_2$. Furthermore, randomly 2-dimensional graphs are determined in [6] and it has been proved that these graphs are odd cycles. Therefore, to complete the characterization, we only need to determine all randomly 3-dimensional graphs.

3 Randomly 3-Dimensional Graphs

In this section, through several lemmas and theorems, we prove that the complete graph K_4 is the unique randomly 3-dimensional graph.

Proposition 3. *If $res(G) = k$, then $\Delta(G) \leq 2^{k-1} + k - 1$.*

Proof. Let $v \in V(G)$ be a vertex with $\deg(v) = \Delta(G)$ and $T = \{v, v_1, v_2, \ldots, v_{k-1}\}$, where $v_1, v_2, \ldots, v_{k-1}$ are neighbors of v. Since $res(G) = k$, T is a resolving set for G. Note that, $d(u, v) = 1$ and $d(u, v_i) \in \{1, 2\}$ for each $u \in N(v) \setminus T$ and each i, $1 \leq i \leq k-1$. Therefore, the maximum number of distinct representations for vertices of $N(v)\setminus T$ is 2^{k-1}. Since T is a resolving set for G, the representations of vertices of $N(v) \setminus T$ are distinct. Thus, $|N(v) \setminus T| \leq 2^{k-1}$ and hence, $\Delta(G) = |N(v)| \leq 2^{k-1} + k - 1$. ■

Lemma 2. *If $res(G) = 3$, then $\Delta(G) \leq 5$.*

Proof. By Proposition 3, $\Delta(G) \leq 6$. Suppose, on the contrary that, there exists a vertex $v \in V(G)$ with $\deg(v) = 6$ and $N(v) = \{x, y, v_1, \ldots, v_4\}$. Since $res(G) = 3$, set $\{v, x, y\}$ is a resolving set for G. Therefore, the representations of vertices $v_1, \ldots, v_4$ with respect to this set are $r_1 = (1, 1, 1)$, $r_2 = (1, 1, 2)$, $r_3 = (1, 2, 1)$, and $r_4 = (1, 2, 2)$. Without loss of generality, we can assume $r(v_i|\{v, x, y\}) = r_i$, for each i, $1 \leq i \leq 4$. Thus, $y \nsim v_2$, $y \nsim v_4$, and $y \sim v_3$.

On the other hand, set $\{v, y, v_3\}$ is a resolving set for G, too. Hence, the representations of vertices x, v_1, v_2, v_4 with respect to this set are r_1, r_2, r_3, r_4 in some order. Therefore, the vertex y has two neighbors and two non-neighbors in $\{x, v_1, v_2, v_4\}$. Since $y \nsim v_2$ and $y \nsim v_4$, the vertices x, v_1 are adjacent to y. Thus, $r(y|\{x, v_1, v_3\}) = (1, 1, 1) = r(v|\{x, v_1, v_3\})$, which contradicts $res(G) = 3$. Hence, $\Delta(G) \leq 5$. ■

Lemma 3. *If $res(G) = 3$ and $v \in V(G)$ is a vertex with $\deg(v) = 5$, then the induced subgraph $\langle N(v)\rangle$ is a cycle C_5.*

Proof. Let $H = \langle N(v)\rangle$. By Theorem D, for each $x \in N(v)$ we have, $|N(x) \cap N(v)| \leq 2$. Therefore, $\Delta(H) \leq 2$, thus, each component of H is a path or a cycle. If the largest component of H has at most three vertices, then there are two vertices $x, y \in N(v)$ which are not adjacent to any vertex in $N(v) \setminus \{x, y\}$. Thus, for each $u \in N(v) \setminus \{x, y\}$, $r(u|\{v, x, y\}) = (1, 2, 2)$, which contradicts the fact that $res(G) = 3$. Therefore, the largest component of H, say H_1, has at least four vertices and the other component has at most one vertex, say $\{x\}$. Let (y_1, y_2, y_3) be a path in H_1. Hence $r(y_1|\{v, x, y_2\}) = (1, 2, 1) = r(y_3|\{v, x, y_2\})$, which is a contradiction. Therefore, $H = C_5$ or $H = P_5$. If $H = P_5 = (y_1, y_2, y_3, y_4, y_5)$, then $r(y_4|\{v, y_1, y_2\}) = (1, 2, 2) = r(y_5|\{v, y_1, y_2\})$, which is impossible. Therefore, $H = C_5$. ■

Lemma 4. *If $res(G) = 3$ and $v \in V(G)$ is a vertex with $\deg(v) = 4$, then the induced subgraph $\langle N(v)\rangle$ is a path P_4.*

Proof. Let $H = \langle N(v)\rangle$. By Theorem D, for each $x \in N(v)$, we have $|N(x) \cap N(v)| \leq 2$. Hence, $\Delta(H) \leq 2$ thus, each component of H is a path or a cycle. If H has more than two components, then it has at least two components with one vertex say $\{x\}$ and $\{y\}$. Thus, $r(u|\{v, x, y\}) = (1, 2, 2)$, for each $u \in N(v) \setminus \{x, y\}$, which contradicts $res(G) = 3$. If H has exactly two components $H_1 = \{x, y\}$ and $H_2 = \{u, w\}$, then $r(u|\{v, x, y\}) = (1, 2, 2) = r(w|\{v, x, y\})$, which is a contradiction. Now, let H has a component with one vertex, say $\{x\}$, and a component contains a path (y_1, y_2, y_3). Consequently, $r(u|\{v, x, y_2\}) = (1, 2, 1)$, for each $u \in N(v) \setminus \{x, y\}$, which is a contradiction. Therefore, $H = C_4$ or $H = P_4$. If $H = C_4 = (y_1, y_2, y_3, y_4, y_1)$, then $r(y_1|\{v, y_2, y_4\}) = (1, 1, 1) = r(y_3|\{v, y_2, y_4\})$, which is impossible. Therefore, $H = P_4$. ■

Proposition 4. *If G is a randomly 3-dimensional graph, then $\Delta(G) \leq 3$.*

Proof. By Lemma 2, $\Delta(G) \leq 5$. If there exists a vertex $v \in V(G)$ with $\deg(v) = 5$, then, by Lemma 3, $\langle N(v) \rangle = C_5$. If $\Gamma_2(v) = \emptyset$, then $G = C_5 \vee K_1$ (the join of graphs C_5 and K_1) and hence, $\beta(G) = 2$, which is a contradiction. Thus, $\Gamma_2(v) \neq \emptyset$. Let $u \in \Gamma_2(v)$. Then u has a neighbor in $N(v)$, say x. Since $\langle N(v) \rangle = C_5$, x has exactly two neighbors in $N(v)$, say x_1, x_2. Therefore, $\deg(x) \geq 4$. By Lemmas 3 and 4, $\langle \{u, v, x_1, x_2\} \rangle = P_4$. Note that, by Theorem D, u has at most two neighbors in $N(v)$. Thus, u is adjacent to exactly one of x_1 and x_2, say x_1. As in Figure 1(a), the set $\{u, v, s\}$ is not a resolving set for G, because $r(x|\{u, v, s\}) = (1, 1, 2) = r(x_1|\{u, v, s\})$. This contradiction implies that $\Delta(G) \leq 4$.

If v is a vertex of degree 4 in G, then by Lemma 4, $\langle N(v) \rangle = P_4$. Let $\langle N(v) \rangle = (x_1, x_2, x_3, x_4)$. If $\Gamma_2(v) = \emptyset$, then $G = P_4 \vee K_1$ and consequently, $\beta(G) = 2$, which is a contradiction. Thus, $\Gamma_2(v) \neq \emptyset$. Let $u \in \Gamma_2(v)$. Then, u has a neighbor in $N(v)$ and by Theorem D, u has at most two neighbors in $N(v)$. If u has only one neighbor in $N(v)$, then by symmetry, we can assume $u \sim x_1$ or $u \sim x_2$. If $u \sim x_2$ and $u \nsim x_1$, then $\deg(x_2) = 4$ and by Lemma 4, $\langle \{u, x_1, x_3, v\} \rangle = P_4$. Therefore, u has two neighbors in $N(v)$, which is a contradiction. If $u \sim x_1$ and $u \nsim x_2$, then $r(v|\{x_1, x_3, u\}) = (1, 1, 2) = r(x_2|\{x_1, x_3, u\})$, which contradicts $res(G) = 3$. Hence, u has exactly two neighbors in $N(v)$. Let $T = N(u) \cap N(v)$. By symmetry, we can assume that T is one of the sets $\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}$, and $\{x_2, x_3\}$. If $T = \{x_1, x_2\}$, then $r(x_1|\{v, x_4, u\}) = (1, 2, 1) = r(x_2|\{v, x_4, u\})$. If $T = \{x_1, x_3\}$, then $r(x_1|\{v, x_2, u\}) = (1, 1, 1) = r(x_3|\{v, x_2, u\})$. If $T = \{x_1, x_4\}$, then $r(v|\{x_1, x_3, u\}) = (1, 1, 2) = r(x_2|\{x_1, x_3, u\})$. These contradictions, imply that $T = \{x_2, x_3\}$. Thus, $|\Gamma_2(v)| = 1$, because each vertex of $\Gamma_2(v)$ is adjacent to both vertices x_2 and x_3 and if $\Gamma_2(v)$ has more than one vertex, then $\deg(x_2) = \deg(x_3) \geq 5$, which is impossible. Now, if $\Gamma_3(v) = \emptyset$, then $\{x_1, x_4\}$ is a resolving set for G, which is a contradiction. Therefore, $\Gamma_3(v) \neq \emptyset$ and hence, u is a cut vertex in G, which contradicts the 2-connectivity of G (Theorem B). Consequently, $\Delta(G) \leq 3$. ∎

Theorem 3. *If G is a randomly 3-dimensional graph, then G is 3-regular.*

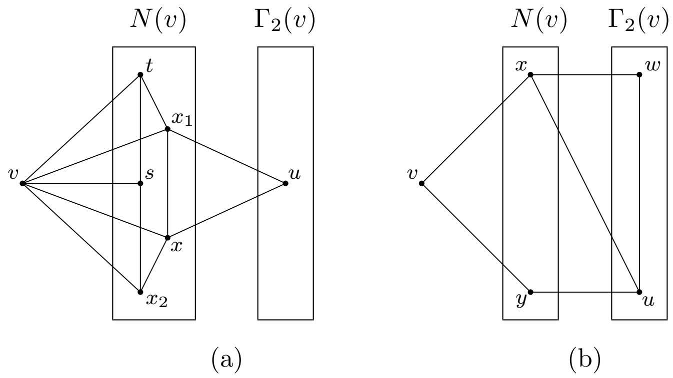

Figure 1: (a) $\Delta(G) = 5$, (b) Neighbors of a vertex of degree 2.

Proof. By Proposition 4, $\Delta(G) \leq 3$ and by Theorem B, $\delta(G) \geq 2$. Suppose that, v is a vertex of degree 2 in G. Let $N(v) = \{x, y\}$. Since $N(v)$ is a separating set of size 2 in G, Theorem C implies that $G \setminus \{v, x, y\}$ is a connected graph and there exists a vertex $u \in V(G) \setminus \{v, x, y\}$ such that $u \sim x$ and $u \sim y$. Note that $G \neq K_n$, because G has a vertex of degree 2 and $\beta(G) = 3$. Thus, by Proposition A, there exists a vertex $w \in V(G)$ such that $w \sim u$ and $w \nsim v$.

If w is neither adjacent to x nor y, then $r(x|\{v, u, w\}) = (1, 1, 2) = r(y|\{v, u, w\})$, which contradicts the fact that $res(G) = 3$. Also, if w is adjacent to both x and y, then $r(x|\{v, u, w\}) = (1, 1, 1) = r(y|\{v, u, w\})$, which is a contradiction. Hence, w is adjacent to exactly one of the vertices x and y, say x. Since $\Delta(G) \leq 3$, the graph in Figure 1(b) is an induced subgraph of G. Clearly, the metric dimension of this subgraph is 2. Therefore, G has at least six vertices.

If $|\Gamma_2(v)| = 2$, then w is a cut vertex in G, because $\Delta(G) \leq 3$. This contradiction implies that there exists a vertex z in $\Gamma_2(v) \setminus \{u, w\}$. Since $\Delta(G) \leq 3$, $z \sim y$. If $z \sim w$, then the graph in Figure 2(a) is an induced subgraph of G with metric dimension 2. In this case, G must have at least seven vertices and consequently, z is a cut vertex in G, which contradicts Theorem B. Hence, $z \nsim w$. By Theorem B, $\deg(z) \geq 2$. Therefore, z has a neighbor in $\Gamma_3(v)$. If there exists a vertex $s \in \Gamma_3(v)$ such that $s \sim z$ and $s \nsim w$, then $r(v|\{y, z, s\}) = (1, 2, 3) = r(u|\{y, z, s\})$, which contradicts $res(G) = 3$. Thus, w is adjacent to all neighbors of z in $\Gamma_3(v)$. Since $\Delta(G) \leq 3$, z has exactly one neighbor in $\Gamma_3(v)$, say t. Hence $\Gamma_3(v) = \{t\}$.

If G has more vertices, then t is a cut vertex in G, which contradicts the 2-connectivity of G. Therefore, G is as in Figure 2(b) and consequently, $\beta(G) = 2$, which is a contradiction. Thus, G does not have any vertex of degree 2. ■

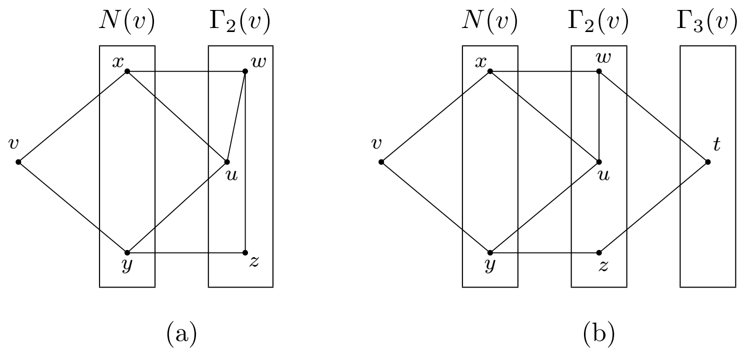

Figure 2: The minimum degree of G is more than 2.

Theorem 4. *If G is a randomly 3-dimensional graph, then G is 3-connected.*

Proof. Suppose, on the contrary, that G is not 3-connected. Therefore, by Theorem B, the connectivity of G is 2. Since G is 3-regular, (by Theorem 4.1.11 in [16],) the edge-connectivity of G is also 2. Thus, there exists a minimum edge cut in G of size 2, say $\{xu, yv\}$. Let H and H_1 be components of $G \setminus \{xu, yv\}$ such that $x, y \in V(H)$ and $u, v \in V(H_1)$. Note that, $x \neq y$ and $u \neq v$, because G is 2-connected. Since G is 3-regular, $|H| \geq 3$ and $|H_1| \geq 3$. Therefore, $\{x, y\}$ is a separating set in G and components of $G \setminus \{x, y\}$ are H_1 and $H_2 = H \setminus \{x, y\}$. Hence, each of the vertices x and y has exactly one neighbor in H_1, u and v, respectively. Since G is 3-regular, x has at most two neighbors in H_2 and u has exactly two neighbors s, t in H_1. Thus, u has a neighbor in H_1 other than v, say s. Therefore, $s \nsim x$ and $s \nsim y$.

If x has two neighbors p, q in H_2, then $r(p|\{x, u, s\}) = (1, 2, 3) = r(q|\{x, u, s\})$, which contradicts $res(G) = 3$. Consequently, x has exactly one neighbor in H_2, say p. Since G is 3-regular, $x \sim y$ and hence, y has exactly one neighbor in H_2. Note that p is not the unique neighbor of y in H_2, because G is 2-connected. Thus, $d(t, p) = 3$ and hence, $r(s|\{u, x, p\}) = (1, 2, 3) = r(t|\{u, x, p\})$, which is impossible. Therefore, G is 3-connected. ■

Proposition 5. *If $G \neq K_4$ is a randomly 3-dimensional graph, then for each $v \in V(G)$, $N(v)$ is an independent set in G.*

Proof. Suppose on the contrary that there exists a vertex $v \in V(G)$, such that $N(v)$ is not an independent set in G. By Theorem 3, $\deg(v) = 3$. Let $N(v) = \{u_1, u_2, u_3\}$. Since $G \neq K_4$, the induced subgraph $\langle N(v)\rangle$ of G has one or two edges. If $\langle N(v)\rangle$ has two edges, then by symmetry, let $u_1 \sim u_2$, $u_2 \sim u_3$ and $u_1 \nsim u_3$. Since G is 3-regular, the set $\{u_1, u_3\}$ is a separating set in G, which contradicts Theorem 4. This argument implies that for each $s \in V(G)$, $\langle N(s)\rangle$ does not have two edges. Hence, $\langle N(v)\rangle$ has one edge, say u_1u_2. Since G is 3-regular, there are exactly four edges between $N(v)$ and $\Gamma_2(v)$. Therefore, $\Gamma_2(v)$ has at most four vertices, because each vertex of $\Gamma_2(v)$ has a neighbor in $N(v)$. On the other hand, 3-regularity of G forces $\Gamma_2(v)$ to have at least two vertices. Thus, one of the following cases can happen.

1. $|\Gamma_2(v)| = 2$. In this case $\Gamma_3(v) = \emptyset$, otherwise $\Gamma_2(v)$ is a separating set of size 2, which is impossible. Consequently, G is as in Figure 3(a). Hence, $\beta(G) = 2$. But, by assumption $\beta(G) = 3$, a contradiction.

2. $|\Gamma_2(v)| = 3$. Let $\Gamma_2(v) = \{x, y, z\}$ and $N(u_3) \cap \Gamma_2(v) = \{y, z\}$. Also, by symmetry, let $u_1 \sim x$, because each vertex of $\Gamma_2(v)$ has a neighbor in $N(v)$. Then the last edge between $N(v)$ and $\Gamma_2(v)$ is one of u_2x, u_2y, and u_2z. But, $u_2x \notin E(G)$, otherwise $\langle N(u_2)\rangle$ has two edges. Thus, by symmetry, we can assume that $u_2y \in E(G)$ and $u_2z \notin E(G)$. Since $res(G) = 3$, we have $y \sim z$, otherwise $r(v|\{u_2, u_3, z\}) = (1, 1, 2) = r(y|\{u_2, u_3, z\})$, which is impossible. For 3-regularity of G, $\Gamma_3(v) \neq \emptyset$. Hence, $\{x, z\}$ is a separating set of size 2 in G, which contradicts Theorem 4.

3. $|\Gamma_2(v)| = 4$. Let $\Gamma_2(v) = \{w, x, y, z\}$ and $u_1 \sim w$, $u_2 \sim x$, $u_3 \sim y$, and $u_3 \sim z$. If $x \nsim y$ and $x \nsim z$, then $d(y, u_2) = 3 = d(z, u_2)$ and it yields $r(y|\{v, u_2, u_3\}) = (2, 3, 1) = r(z|\{v, u_2, u_3\})$. Therefore, G has at least one of the edges xy and xz. If G has both xy and xz, then $r(y|\{v, x, u_3\}) = r(z|\{v, x, u_3\})$. Thus, G has exactly one of the edges xy and xz, say xy. In the same way, G has exactly one of the edges wy and wz. If $w \sim y$, then $r(x|\{v, u_3, y\}) = (2, 2, 1) = r(w|\{v, u_3, y\})$. Hence, $w \nsim y$ and $w \sim z$. Note that, $x \nsim w$, otherwise $r(u_2|\{u_1, x, u_3\}) = (1, 1, 2) = r(w|\{u_1, x, u_3\})$. Therefore, $N(w) \cap [\Gamma_1(v) \cup \Gamma_2(v)] = \{u_1, z\}$. Since G is 3-regular,

$\Gamma_3(v) \neq \emptyset$. If $z \sim y$, then $\{w, x\}$ is a separating set in G which is impossible. Thus, z has a neighbor in $\Gamma_3(v)$, say u. If $u \nsim w$, then $d(w, u) = 2 = d(u_3, u)$ which implies that $r(u_3|\{u_2, z, u\}) = (2, 1, 2) = r(w|\{u_2, z, u\})$. Hence, $u \sim w$ and it yields $r(w|\{u, v, x\}) = r(z|\{u, v, x\})$. Consequently, $N(v)$ is an independent set in G. ■

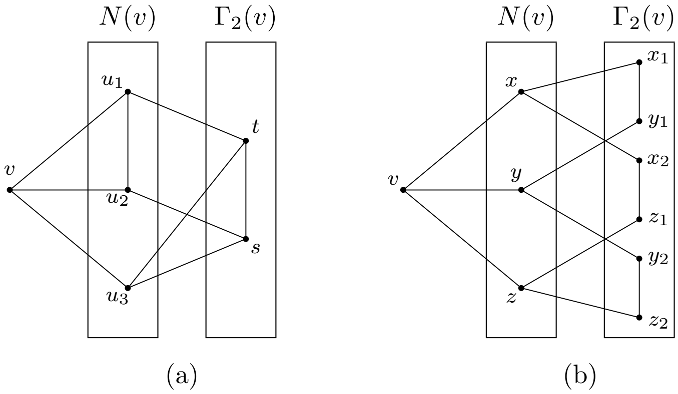

Figure 3: Two graphs with metric dimension 2.

Theorem 5. *If G is a randomly 3-dimensional graph, then $G = K_4$.*

Proof. Suppose on the contrary that G is a randomly 3-dimensional graph and $G \neq K_4$. Let $v \in V(G)$ be an arbitrary fixed vertex and $N(v) = \{x, y, z\}$. By Proposition 5, $N(v)$ is an independent set in G. Since G is 3-regular, there are six edges between $N(v)$ and $\Gamma_2(v)$. If a vertex $a \in \Gamma_2(v)$ is adjacent to x and y, then $r(x|\{v, a, z\}) = (1, 1, 2) = r(y|\{v, a, z\})$, which is impossible. Therefore, by symmetry, each vertex of $\Gamma_2(v)$ has exactly one neighbor in $N(v)$ and hence $\Gamma_2(v)$ has exactly six vertices. If there exists a vertex $a \in \Gamma_2(v)$ with no neighbor in $\Gamma_2(v)$, then by symmetry, let $a \sim z$. Thus, $r(x|\{v, z, a\}) = (1, 2, 3) = r(y|\{v, z, a\})$. Also, if there exists a vertex $a \in \Gamma_2(v)$ with two neighbors b and c in $\Gamma_2(v)$, by symmetry, let $a \sim z$, $b \nsim z$ and $c \nsim z$. Then, $r(b|\{v, z, a\}) = (2, 2, 1) = r(c|\{v, z, a\})$. These contradictions imply that $\Gamma_2(v)$ is a matching in G. Since all neighbors of each vertex of G constitute an independent set in G, the induced subgraph $\langle \{v\} \cup N(v) \cup \Gamma_2(v) \rangle$ of G is as in Figure 3(b). Since G is 3-regular, $\Gamma_3(v) \neq \emptyset$ and each vertex of $\Gamma_2(v)$ has one neighbor in $\Gamma_3(v)$. Let $u \in \Gamma_3(v)$

be the neighbor of x_1. Thus, $y_1 \nsim u$. If y_1 and z_2 have no common neighbor in $\Gamma_3(v)$, then $r(x|\{x_1, u, z_2\}) = (1, 2, 3) = r(y_1|\{x_1, u, z_2\})$. Therefore, y_1 and z_2 have a common neighbor in $\Gamma_3(v)$, say w. Consequently, $r(y|\{v, x, w\}) = (1, 2, 2) = r(z|\{v, x, w\})$. This contradiction implies that $G = K_4$. ■

The next corollary characterizes all randomly k-dimensional graphs.

Corollary 1. *Let G be a graph with $\beta(G) = k > 1$. Then, G is a randomly k-dimensional graph if and only if G is a complete graph K_{k+1} or an odd cycle.*

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