Error Propagation Solutions

Throughout all of these solutions we will be quoting equations from the front and back pages of "*Measurements and their Uncertainties",* Ifan G. Hughes and Thomas P.A. Hase, without proof. For more details on these equations, and where they come from, please refer to chapter 4 of the book. We will also adopt their notation for *Z, A, B* and *k*.

Question 1

a) We know from Newton's second law of motion that the force due to gravity is given by $F = mg$. We are assuming a negligible error in g, and therefore to obtain the error in *F*, we simply multiply the error in m by a factor of g. This follows from the rule:

$$Z = kA \Rightarrow \alpha_Z = |k|\alpha_A$$

Using this formula, it follows that:

$$\alpha_F = g\alpha_m$$
$$\Rightarrow \alpha_F = 9.81 \times (0.01 \times 10^{-3}) = 0.1 \times 10^{-3}\ \mathrm{N}\ (1.\mathrm{S.F})$$

This value, and a simple calculation of the force *F*, gives us the final answer of

$$F = (29.4 \pm 0.1) \times 10^{-3}\ \mathrm{N}$$

b) Now we must take into account the error in *g.* Since we are multiplying two numbers with errors together, we need to add the **fractional** errors in quadrature. This is shown below.

$$Z = AB \Rightarrow \frac{\alpha_Z}{Z} = \sqrt{\left(\frac{\alpha_A}{A}\right)^2 + \left(\frac{\alpha_B}{B}\right)^2}$$

Therefore, in this case it follows:

$$\alpha_F = F \times \sqrt{\left(\frac{\alpha_m}{m}\right)^2 + \left(\frac{\alpha_g}{g}\right)^2}$$
$$\Rightarrow \alpha_F = 29.43 \times \sqrt{\left(\frac{0.1}{3}\right)^2 + \left(\frac{0.02}{9.81}\right)^2} = 0.1 \times 10^{-3}\ \mathrm{N}$$

To 1.S.F, there is no change in the error in *F*. This is because if you consider the fractional error in *m,* we find it is of the order of 1%, whereas the fractional error in g is 0.1%. Given that we usually are only interested in the first significant figure, the g term is an order of magnitude less and therefore can be ignored. This is a big time saver when solving problems like this!
Also, notice how in the m contribution to the error in *F* the order of magnitude has been ignored (i.e. the "10^{-3}" term). This is because we are dividing the top and bottom by the same value, and therefore doesn't need to be taken into account. This is very useful, especially when dealing with strange units which can be ignored.

c) The equation for calculating the net force is given by:

$$F_{net} = F - F_{res}$$

Therefore, we add the errors in quadrature, **not** their fractional errors.

$$\alpha_{Fnet} = \sqrt{(\alpha_F)^2 + (\alpha_{Fres})^2}$$
$$\Rightarrow \alpha_{Fnet} = \sqrt{(0.1 \times 10^{-3})^2 + (0.2 \times 10^{-3})^2} = 0.2 \times 10^{-3} \text{ N}$$

Notice this time we do have to take into account the order of magnitude, given that these are not fractional errors.

Question 2

Before we solve this problem, it is important to note that at no point in this question is an error calculation explicitly asked for. However, you should **always** assume that an error calculation is required if errors are given or you've calculated them from other sections of the experiment. Remember, calculating *F* or *r* without errors is not a complete answer to this problem.

a) Firstly we need to compute the value of *F*, this is a straightforward calculation shown below.

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} = \frac{1.6 \times 10^{-6} \times 2.5 \times 10^{-6}}{4\pi \times 8.85 \times 10^{-12} \times (1.5)^2} = 0.01598 \ldots \text{N}$$

Now we must compute the error in *F*. To solve a problem like this, it's easiest to work out the errors of the smaller components of the equation first. So consider the error in r^2 first.

$$\alpha_{r^2} = 2r\alpha_r$$

This comes from the standard rules of errors of variables raised to powers. Now we know the error in r^2, notice how we now essentially have an equation of the form

$$Z = \frac{AB}{C}$$

It follows then that

$$\frac{\alpha_F}{F} = \sqrt{\left(\frac{\alpha_{Q_1}}{Q_1}\right)^2 + \left(\frac{\alpha_{Q_2}}{Q_2}\right)^2 + \left(2\frac{\alpha_r}{r}\right)^2}$$

Notice how the factor of "2" from the error in *r* is *inside* the bracket, not outside. This is a common mistake students make, we are squaring the entire error contribution from r^2 **including** the factor of 2.

Also, there is no need to divide the error by the prefactor "$4\pi\epsilon_0$". This term is taken into account by the fact that we are dealing with fractional errors, and so it is included in the *F* term. We are therefore left with

$$\frac{\alpha_F}{F} = \sqrt{\left(\frac{\alpha_{Q_1}}{Q_1}\right)^2 + \left(\frac{\alpha_{Q_2}}{Q_2}\right)^2 + 4\left(\frac{\alpha_r}{r}\right)^2}$$

as our final equation for the error in *F*. Plugging in the numbers gives an error of $5\ \times 10^{-3}$ N. Therefore the final answer for *F* is

$$(16 \pm 5) \times 10^{-3}\ \text{N}$$

b) We must first rearrange the equation so that it is in terms of *r*.

$$r = \sqrt{\frac{Q_1 Q_2}{4\pi\epsilon_0 F}}$$

Similarly to above, we need to perform a straightforward calculation of to obtain a value for *r.* Plugging in the numbers gives *r* to be 0.1250… m.

To calculate the error, we need to follow similar steps to above. Notice how the terms inside the square root consists of just multiplication and division of variables. Therefore, define a variable *x* such that

$$x = \frac{Q_1 Q_2}{4\pi\epsilon_0 F}$$

Therefore, the error in *x*, similar to above, is given below.

$$\frac{\alpha_x}{x} = \sqrt{\left(\frac{\alpha_{Q_1}}{Q_1}\right)^2 + \left(\frac{\alpha_{Q_2}}{Q_2}\right)^2 + \left(\frac{\alpha_F}{F}\right)^2}$$

Using the power relation for errors, we can define the error of α_r in terms of α_x.

$$\alpha_r = \frac{1}{2}\alpha_x$$

Therefore we can combine the two equations above to give.

$$\alpha_r = \frac{1}{2}\,x\sqrt{\left(\frac{\alpha_{Q_1}}{Q_1}\right)^2 + \left(\frac{\alpha_{Q_2}}{Q_2}\right)^2 + \left(\frac{\alpha_F}{F}\right)^2}$$

Or

$$\alpha_r = \frac{1}{2}\,\frac{Q_1 Q_2}{4\pi\epsilon_0 F}\sqrt{\left(\frac{\alpha_{Q_1}}{Q_1}\right)^2 + \left(\frac{\alpha_{Q_2}}{Q_2}\right)^2 + \left(\frac{\alpha_F}{F}\right)^2}$$

Plugging in the numbers to this equation gives

$$\alpha_r = \frac{1}{2}\,\frac{1.6\times 10^{-6}\times 2.5\times 10^{-6}}{4\pi\times 8.85\times 10^{-12}\times 2.3}\sqrt{\left(\frac{0.3}{1.6}\right)^2 + \left(\frac{0.1}{2.5}\right)^2 + \left(\frac{0.5}{2.3}\right)^2}$$

$$\alpha_r = 2\times 10^{-3}\text{ m}$$

Therefore, the final answer for *r* is

$$r = (125 \pm 2)\text{ mm}$$

Question 3

a) You should be able to recall that the half life and the decay rate are related by the following equation

$$\lambda = \frac{\ln 2}{T_{\frac{1}{2}}}$$

Therefore the decay rate, without error, is simply

$$\lambda = \frac{\ln 2}{1.41 \times 10^{17}} = 4.9159 \ldots \times 10^{-18}\ \mathrm{s}^{-1}$$

The error in this term can be calculated by considering the following rule

$$Z = k\,\frac{B}{A} \Rightarrow \frac{\alpha_Z}{Z} = \sqrt{\left(\frac{\alpha_A}{A}\right)^2 + \left(\frac{\alpha_B}{B}\right)^2}$$

Since we have no "B" term, set $\alpha_B = 0$ and it follows that

$$\frac{\alpha_\lambda}{\lambda} = \frac{\alpha_{T_{1/2}}}{T_{\frac{1}{2}}}$$

From this equation, and a substitution for λ, we can deduce the following relation

$$\alpha_\lambda = \frac{\ln 2}{(T_{\frac{1}{2}})^2}\alpha_{T_{\frac{1}{2}}}$$

$$\alpha_\lambda = \frac{\ln 2}{(1.41 \times 10^{17})^2} \times 0.01 \times 10^{17} = 0.03 \times 10^{-18}\ \mathrm{s}^{-1}$$

Giving us the final answer of $\lambda = (4.92 \pm 0.03) \times 10^{-18}\ \mathrm{s}^{-1}$

b) Like before, we need to firstly calculate the numerical value for *N(t)* before considering its error.

$$N(t) = N_0 \exp(-\lambda t)$$
$$N(t) = (1.6 \times 10^{28}) \exp(-4.92 \times 10^{-18} \times 2.0 \times 10^{18}) = 8.5939 \ldots \times 10^{23}$$

To calculate the error of *N(t)*, it's better to work out the error inside the exponent and then work outwards. So, define *x* such that

$$x = \lambda t$$

The error in *x* is given by $\alpha_x = t\alpha_\lambda$. Then, define another variable, *y*, such that

$$y = \exp(-x)$$

The error is

$$\alpha_y = \exp(-x)\alpha_x$$
$$\Rightarrow \alpha_y = \exp(-\lambda t) \times t\alpha_\lambda$$

Where we have simply substituted our definitions for *x* and *y* into the equation. Notice how the equation for *N(t)* essentially boils down to a product of *N(t)* and *y*. This means we can add the fractional errors for *y* and *N(t)* in quadrature.

$$\frac{\alpha_{N(t)}}{N(t)} = \sqrt{\left(\frac{\alpha_y}{y}\right)^2 + \left(\frac{\alpha_{N_0}}{N_0}\right)^2}$$

Substituting all of the above equations together, we get

$$\alpha_{N(t)} = N(t)\sqrt{\left(\frac{\exp(-\lambda t)\, t\alpha_\lambda}{\exp(-\lambda t)}\right)^2 + \left(\frac{\alpha_{N_0}}{N_0}\right)^2}$$

If you're not comfortable combining all of these together algebraically, feel free to work out the numerical values for each element individually before substituting it into the equation for $\alpha_{N(t)}$.

Plugging in the numbers, we get an error of 1×10^{23} atoms. So our final answer for *N(t)* is

$$N(t) = (9 \pm 1) \times 10^{23} \text{ atoms}$$

c) The experimental value lies within one error bar of the theoretical value for *N(t)*. Therefore, we can conclude that there is a reasonable fit between experiment and theory.

Note however, that this does not imply that the theory is "correct" or "right". The only thing we can conclude is that this theory's prediction matches the results of this experiment. A better experiment could be run at a later time which contradicts the theory; hence we cannot say it is absolutely "correct".

A more complete conclusion of whether the model fits the data requires χ^2 data analysis, which you will come across in level 2.

Question 4

This problem involves many steps to solve for the error in R. Thankfully, we can make a few jumps in logic that'll reduce the number of steps we have to take.

- The error in $\theta_i - \theta_t$ and $\theta_i + \theta_t$ are the same, therefore define x such that $x = \theta_i \pm \theta_t$ and therefore $\alpha_x = \sqrt{(\alpha_{\theta_i})^2 + (\alpha_{\theta_t})^2}$ in both cases.
- Define $y^{\pm} = \tan(\theta_i \pm \theta_t)$, therefore $\alpha_{y^{\pm}} = (1 + tan^2(\theta_i \pm \theta_t))\alpha_x$ using the standard rules.
- Since we are considering the square of y^+ and y^-, we must multiply the fractional errors in α_{y^+} and α_{y^-} by 2.
- Using the normal product rules, this gives us

$$\alpha_R = R\sqrt{\left(2\frac{\alpha_{y^+}}{y^+}\right)^2 + \left(2\frac{\alpha_{y^-}}{y^-}\right)^2}$$

$$\alpha_R = 2R\sqrt{\left(\frac{\alpha_{y^+}}{y^+}\right)^2 + \left(\frac{\alpha_{y^-}}{y^-}\right)^2}$$

Following through the calculations, you should get

$$y^+ = 5.4 \pm 0.1$$

$$y^- = (185 \pm 4) \times 10^{-3}$$

$$R = (1.18 \pm 0.07) \times 10^{-3}$$

Question 5

If we rearrange Snell's law for θ_r, we can see that we're going to come across a problem

$$\theta_r = arcsin\left(\frac{sin\ \theta_i}{n}\right)$$

How do we work out the error in an arcsin function? We could take a partial differential approach, but that would be difficult. We can instead consider a functional approach.

Firstly, define *x* such that

$$x = \frac{sin\ \theta_i}{n}$$

We can combine these errors in the usual way, giving us the result

$$\alpha_x = \frac{sin\ \theta_i}{n}\sqrt{\left(\frac{\alpha_n}{n}\right)^2 + \left(\frac{\cos(\theta_i)}{\sin(\theta_i)}\alpha_{\theta_i}\right)^2}$$

From this, we can estimate the error in α_{θ_r} by considering the following

$$\alpha_{\theta_r}^{\pm} = |\arcsin(x) - \arcsin(x \pm \alpha_x)|$$

It's important to note that because arcsin is non-linear, you'll get different sized error bars in the positive and negative directions. Using the above, we can calculate that

$$x = (274 \pm 2) \times 10^{-3}$$

(Note that be careful with degrees and radians, α_{θ_i} must be in radians!)

Using this value for *x*, we can show that

$$\alpha_{\theta_r}^{+} = 0.1°$$

$$\alpha_{\theta_r}^{-} = 0.1°$$

Therefore the final answer for θ_r is

$$\theta_r = (15.9 \pm 0.1)°$$